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Constructing Surfaces with (1/(k-2)^2)(1,k-3) Singularities

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CONSTRUCTING SURFACES WITH $\frac{1}{(k-2)^2}(1, k-3)$ SINGULARITIES

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ABSTRACT. We develop a procedure to construct complex algebraic surfaces which are stable, minimal, and of general type, possessing a T-singularity of the form $\frac{1}{(k-2)^2}(1, k-3)$.

CONTENTS

1. INTRODUCTION

Moduli theory has a rich tradition in algebraic geometry. In this paper, the example of interest is the quasi-projective moduli space of smooth minimal surfaces of general type constructed by Gieseker in [6]. In particular, Gieseker proved the following theorem:

Theorem 1.1. [6] *Fix positive integers K^2 and χ. Then, there exists a quasi-projective variety, denoted $M_{K^2,\chi}$, which is a moduli space for smooth surfaces, X, minimal and of general type over $\mathbb{C}$ with fixed $K_X^2 = K^2$ and $\chi(\mathcal{O}_X) = \chi$.*

Analogous to the work of Deligne and Mumford in compactifying M_g (see [4]), the moduli space of smooth curves of genus g, the combined efforts of Kollár, Shepherd-Barron, and Alexeev in [11] and [1] yielded a compactification of Gieseker's moduli spaces.

Theorem 1.2. [11], [1] *Fix positive integers K^2 and χ. There exists a compactification, $\overline{M}_{K^2,\chi}$, of the moduli space $M_{K^2,\chi}$.*

The surfaces parameterized by $\overline{M}_{K^2,\chi}$ are called *stable surfaces*, which are allowed to have mild surface singularities. One kind of surface singularity appearing on stable surfaces are *T-singularities*, which are the focus of this paper. Recall that a *cyclic quotient singularity* of type $\frac{1}{n}(1, a)$ is the singularity obtained from the quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}/n\mathbb{Z}$, with action given by the matrix

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}$$

where $\varepsilon = \exp(\frac{2\pi i}{n})$ is a primitive nth root of unity and a is coprime to n. Now a *T-singularity* is a cyclic quotient singularity of the form

$$\frac{1}{dn^2}(1, dna-1),$$

where n, d, and a are integers with a coprime to n. This will all be revisited in Section 2.

The aim of this paper is to provide a procedure which produces a complex algebraic surface, minimal and of general type, with a T-singularity of the form $\frac{1}{(k-2)^2}(1, k-3)$, where $k \geq 5$. We begin the body of the paper with Section 2 by introducing the key tools and facts needed to prove that our construction works. Section 3 is comprised of our main result, computations related to examples of surfaces containing the T-singularities $\frac{1}{16}(1,3)$ and $\frac{1}{25}(1,4)$, and a connection to a theorem in [10].

2. Background

Every algebraic variety considered is projective and over $\mathbb{C}$. We now recall notation and definitions which are freely used in the sequel. Let X be a smooth algebraic surface.

K_X is the *canonical class* of X.
$p_g(X) = h^0(X, \omega_X) = h^2(X, \mathcal{O}_X)$ is the *geometric genus* of X.
$q(X) = h^1(X, \mathcal{O}_X)$ is the *arithmetic genus* of X.
$\kappa(X)$ is the *Kodaira dimension* of X.
$\chi(X) = \chi(\mathcal{O}_X) = 1 - q + p_g$ is the *Euler characteristic* of X.

The following are fundamental results from algebraic geometry which are frequently used to compute the invariants above.

Theorem 2.1 (Riemann-Roch for Surfaces)**.** *Let X be a smooth algebraic surface, let $\mathcal{L}$ be an invertible sheaf on X, and let D be the divisor associated to $\mathcal{L}$ so that $\mathcal{O}_X(D) \simeq \mathcal{L}$. Then*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X).$$

Proof. This is Theorem I.12 from [2]. □

Theorem 2.2 (Kodaira Vanishing)**.** *Let X be a smooth variety of dimension n and let $\mathcal{L}$ be an ample invertible sheaf on X. Then*

$$H^p(X, \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^q_X) = 0 \ \textit{for} \ \ p + q > n.$$

In particular, for X a smooth algebraic surface, we have

$$H^p(X, \mathcal{L} \otimes \omega_X) = 0 \ \textit{for} \ \ p > 0.$$

Proof. This is Theorem 2.1.3 from [5]. □

Proposition 2.3 (Adjunction Formula)**.** *Let X be a smooth algebraic surface and let C be a smooth curve on X. Then*

$$(C + K_X) \cdot C = 2g(C) - 2,$$

where $g(C)$ is the genus of the curve C.

Proof. This is Proposition 1.5 from Chapter V of [7]. □

Theorem 2.4 (Nakai-Moishezon Criterion). *Let X be a smooth algebraic surface. A divisor D on X is ample if and only if $D^2 > 0$ and for every irreducible curve $C \subseteq X$, the product $D \cdot C > 0$.*

Proof. This is Theorem 1.10 from Chapter V of [7]. □

Proposition 2.5. *Let X be a smooth surface. Let C be an effective divisor on X, let $p \in X$ be a point of multiplicity m on C, and let $\pi : \tilde{X} \to X$ be the blowup of X at p. Then*

$$\pi^* C = \tilde{C} + mE,$$

where $\tilde{C}$ is the strict transform of C and E is the exceptional divisor.

Proof. This is Proposition 3.6 from Chapter V of [7]. □

Theorem 2.6 (Enriques-Kodaira Classification). *Every complex algebraic surface has a minimal model in exactly one of the following ten classes. This minimal model is unique, unless the surface is rational.*

TABLE 1. Minimal surface classes

Class of X	$\kappa(X)$	K_X^2	$p_g(X)$
1) rational		8 *or* 9	
2) class VII	$-\infty$	≥ 0	0
3) ruled of genus $g \geq 1$		$8(1-g)$	
4) Enriques			0
5) bi-elliptic			0
6a) Kodaira (primary)	0	0	1
6b) Kodaira (secondary)			0
7) K3-surfaces			1
8) tori			1
9) properly elliptic	1	0	≥ 0
10) general type	2	> 0	≥ 0

For $n \geq 0$, the n-th Hirzebruch surface is

$$\mathbb{F}_n := \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)\right),$$

the projectivization of the rank two vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ over $\mathbb{P}^1$. Hirzebruch surfaces are examples of geometrically ruled surfaces over $\mathbb{P}^1$ which happen to be minimal for $n \neq 1$, as we will see below. More generally, given a geometrically ruled surface $p : S \to C$, there exists a rank two vector bundle $\mathcal{E}$ such that S and $\mathbb{P}(\mathcal{E})$ are isomorphic as projective bundles over C (Proposition III.7 of [2]). The Hirzebruch surfaces have easily described Picard groups and canonical classes, making intersection products simple to calculate. The following theorem is standard (see [2] and [3]). Nonetheless, we provide a proof to indicate how some of the fundamental tools of algebraic geometry (e.g. homological algebra and sheaf cohomology) are used in practice.

Proposition 2.7. *Let n be an integer with $n \geq 0$. Then*

(1) $\mathrm{Pic}(\mathbb{F}_n) \simeq \mathbb{Z}h \oplus \mathbb{Z}f$ *where h corresponds to $\mathcal{O}_{\mathbb{F}_n}(1)$ and f is the class of a fiber over $\mathbb{P}^1$. Furthermore, $h^2 = n$, $h \cdot f = 1$, and $f^2 = 0$.*
(2) The canonical divisor $K_{\mathbb{F}_n}$ is linearly equivalent to $-2h + (n-2)f$.

Proof. Proposition III.18 from [2] tells us that for a geometrically ruled surface $p : S = \mathbb{P}(\mathcal{E}) \to C$, where h is the class of $\mathcal{O}_S(1)$ in $\mathrm{Pic}(S)$, the following hold

(i) $\mathrm{Pic}(S) \simeq p^* (\mathrm{Pic}(C)) \oplus \mathbb{Z}h$;
(ii) $h^2 = \deg(\mathcal{E})$; and
(iii) $K_S = -2h + (\deg(\mathcal{E}) + 2g(C) - 2) f$ in $\mathrm{Pic}(S)$.

First, we prove (1). The morphism $\mathbb{F}_n \to \mathbb{P}^1$ induces a pullback $p^* : \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(\mathbb{F}_n)$. Because $\mathrm{Pic}(\mathbb{P}^1)$ is the free abelian group generated by the class of a point, $p^* \left(\mathrm{Pic}(\mathbb{P}^1)\right)$ is the free abelian subgroup of $\mathrm{Pic}(\mathbb{F}_n)$ generated by the class of a fiber. Denote this class by f. Thus, from (i) it follows that $\mathrm{Pic}(\mathbb{F}_n) \simeq \mathbb{Z}h \oplus \mathbb{Z}f$.

By Corollary III.12 from [2], the degree of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ is given by

$$\deg(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = \chi(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) - 2\chi(\mathcal{O}_{\mathbb{P}^1}). \tag{2.8}$$

Thus, to determine the left hand side we need only compute the Euler characteristic of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. Because $\mathbb{P}^1$ has genus zero and is one-dimensional, $\chi(\mathcal{O}_{\mathbb{P}^1}) = 1$. Consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0 \tag{2.9}$$

given by the maps $\alpha : s \mapsto (s, 0)$ and $\beta : (s,t) \mapsto t$. Recall that $\mathbb{P}^1$ is one-dimensional and genus zero, so we have $h^i(\mathcal{O}_{\mathbb{P}^1}) = 0$ for all $i \geq 1$. Now, because $\mathcal{O}_{\mathbb{P}^1}(n+2)$ is ample and $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, by the Kodaira vanishing theorem

$$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \simeq H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n+2) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$$

for $i \geq 1$. By the long exact sequence associated to (2.9), it follows that for $i \geq 1$ we have $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = 0$. As a consequence of this, the sequence

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \to 0$$

is exact, and furthermore, $\chi(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. For $n \geq 0$, the $\mathbb{C}$-vector space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ consists of polynomials of degree n on $\mathbb{P}^1$, hence has $\{x^i y^{n-i}\}_{i=0}^n$ as a basis. Thus, $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = n+1$, so

$$\chi(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = n + 2.$$

Thus, the left hand side of (2.8) is n, because $2\chi(\mathcal{O}_{\mathbb{P}^1}) = 2$. Substitute n for $\deg(\mathcal{E})$ in (ii) and (1) is proved.

To prove (2), we substitute n for $\deg(\mathcal{E})$ and 0 for $g(C)$ into the equation given in (iii). □

One interesting consequence of Proposition 2.7 which we prove below is that, on $\mathbb{F}_n$, there is a unique irreducible rational curve with negative self-intersection. Furthermore, this curve has self-intersection equal to $-n$. From this, it will follow that $\mathbb{F}_n$ is minimal for $n \neq 1$ and non-minimal for $n = 1$.

Proposition 2.10. *Let $n \geq 0$ be an integer. On $\mathbb{F}_n$, there exists a unique irreducible curve D with negative self-intersection. Furthermore, $D^2 = -n$.*

Proof. Let D be a curve linearly equivalent to $h - nf$. In order to show that D exists, it suffices to prove that the linear system $|h - nf|$ has nonzero dimension. In particular, one must compute $h^0(\mathcal{O}_{\mathbb{F}_n}(h - nf))$. From this, it follows that $D^2 = (h - nf)^2 = -n$.

Suppose that $n \neq 1$. Let $C \neq D$ be any irreducible curve on $\mathbb{F}_n$, so in the Picard group we can write $C = \alpha h + \beta f$ for some $\alpha, \beta \in \mathbb{Z}$, where at least one of α, β is nonzero. As $C \cdot f \geq 0$, it follows that $\alpha h \cdot f \geq 0$, so that $\alpha \geq 0$. Similarly, since $D = h - nf$ is an irreducible curve on $\mathbb{F}_n$, we have $C \cdot D \geq 0$, which means that $(\alpha h + \beta f) \cdot (h - nf) \geq 0$ which is true if and only if $\beta h \cdot f = \beta \geq 0$. Thus, $C^2 = \alpha^2 n + 2\alpha\beta \geq 0$, so there can be no irreducible curves on $\mathbb{F}_n$ with $C^2 = -1$. It follows that for $n \neq 1$ the surface $\mathbb{F}_n$ is minimal. □

Corollary 2.11. *For $n \neq 1$, we have*

(1) $\kappa(\mathbb{F}_n) = -\infty$
(2) $p_g(\mathbb{F}_n) = 0$*; and*
(3) $K^2_{\mathbb{F}_n} = 8$.

Proof. As $\mathbb{F}_n$ is rational and minimal for $n \neq 1$, (1) and (2) follow immediately from the Enriques-Kodaira classification theorem. From the computation

$$\begin{aligned} K^2_{\mathbb{F}_n} &= (-2h + (n-2)f)^2 \\ &= 4n - 4(n-2) \\ &= 8, \end{aligned}$$

(3) follows. □

2.1. **Singularities.** Recall from the introduction that the moduli spaces $\overline{M}_{K^2,\chi}$ parameterize *stable surfaces*. The types of singularities permitted on stable surfaces are *semi log canonical*. Here we provide a definition of stable surfaces but omit the definition of semi log canonical (this can be found in [11]).

Definition 2.12. A *stable surface* is a reduced Cohen-Macaulay projective surface X over $\mathbb{C}$ such that X has semi log canonical singularities and the dualizing sheaf ω_X is ample.

We recall the definitions of cyclic quotient singularities and T-singularities.

Definition 2.13. A *cyclic quotient singularity* of type $\frac{1}{n}(1,a)$ is a singularity obtained from the quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}/n\mathbb{Z}$, with action given by the matrix

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}$$

where $\varepsilon = \exp(\frac{2\pi i}{n})$ is a primitive nth root of unity and a is coprime to n.

A *T-singularity* is a cyclic quotient singularity of the form

$$\frac{1}{dn^2}(1, dna - 1),$$

where n, d, and a are integers such that a is coprime to n.

By Proposition 3.11 of [11], these singularities are determined by the exceptional curves obtained from the minimal resolution of a surface with such a T-singularity. In this situation, the exceptional divisors form what are called Hirzebruch-Jung strings. In fact, every cyclic quotient singularity can be realized as a contraction of a Hirzebruch-Jung string, and vice versa.

Definition 2.14. Let X be a smooth algebraic surface. A *Hirzebruch-Jung string* on X of length r is a union $C = \bigcup_{i=1}^{r} C_i$ of smooth rational curves such that

$$\begin{aligned} C_i^2 &\le -2 \text{ for all } i, \\ C_i \cdot C_j &= 1 \text{ if } |i-j| = 1, \text{ and} \\ C_i \cdot C_j &= 0 \text{ if } |i-j| \ge 2. \end{aligned}$$

The notation $[b_1, \ldots, b_r]$ is used to denote a Hirzebruch-Jung string $C = \bigcup_{i=1}^r C_i$ where $-b_i = C_i^2$.

The Hirzebruch-Jung string $[b_1, \ldots, b_r]$ is the exceptional divisor of the minimal resolution of a surface Y containing a $\frac{1}{n}(1, a)$ singularity, where

$$\frac{n}{a} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$$

Example 2.15. Let $k \ge 5$ be an integer. Then the string $[k, 2, \ldots, 2]$ of length $k-3$ is a Hirzebruch-Jung string, corresponding to a T-singularity. To determine which T-singularity it corresponds to, we compute the value of

$$k - \cfrac{1}{2 - \cfrac{1}{\ddots - \cfrac{1}{2}}}$$

Because the chain has length $k-3$,

$$\cfrac{1}{2 - \cfrac{1}{\ddots - \cfrac{1}{2}}} = \frac{k-4}{k-3},$$

implying that

$$k - \cfrac{1}{2 - \cfrac{1}{\ddots - \cfrac{1}{2}}} = k - \frac{k-4}{k-3} = \frac{(k-2)^2}{k-3}.$$

Thus, the chain $[k, 2, \ldots, 2]$ of length $k-3$ is the exceptional divisor of a minimal resolution of the T-singularity $\frac{1}{(k-2)^2}(1, k-3)$.

We have been implicitly referencing the fact that T-singularities are semi log canonical, so here we make this fact explicit.

Proposition 2.16 ([8] and [11])**.** *Let X be a smooth algebraic surface, and let $E \subseteq X$ be the Hirzebruch-Jung string $[b_1, \ldots, b_r]$ corresponding to the singularity $\frac{1}{n}(1, a)$. If $f : X \to Y$ is the contraction of E to the point $p \in Y$, then Y has a $\frac{1}{n}(1, a)$ singularity at p, and this singularity is log canonical, hence semi log canonical.*

Proof. This is Remark 4.9 from [8] combined with part (ii) of Theorem 4.21 from [11]. □

2.2. **Mappings of Surfaces.**

Construction 2.17. The following construction is described in the discussion preceding Theorem 7.1 from Chapter III of [3]. Let X be a normal surface and Y a smooth surface, both connected, and let $\gamma : X \to Y$ be a double cover, branched over a divisor $B \subseteq Y$. Assume that B has at worst ADE singularities. Because Y is assumed smooth, the singularities of X must necessarily lie over the singularities of B, and can be resolved via the following procedure. Let μ_y be the multiplicity of $y \in B$ and let $\sigma_1 : Y_1 \to Y$ be the simultaneous blowup of all singular points $y \in B$. If $E_y = \sigma^{-1}(\{y\})$ is the exceptional curve over y, then $\sigma^*(B) = \overline{B} + \sum \mu_y E_y$ is the total transform of B. Let X_1 be the double cover of Y_1, branched over

$$B_1 = \overline{B} + \sum_{\mu_y \text{ odd}} E_y.$$

Unless B_1 is nonsingular we repeat this construction, replacing B with B_1 to obtain B_2 and so on. Since the new branch curves $B_1, B_2, \ldots,$ are contained in the total transforms of B, Theorem 7.2 of Chapter II of [3] implies that after finitely many steps we obtain a smooth curve B_k, hence X_k is a resolution of singularities for X. In the following theorem we will denote this smooth curve B_k by B_{res}. We call this divisor the *resolved branch divisor of B*.

Theorem 2.18. [3] *Let $\gamma : X \to Y$ be a double cover with X normal and Y nonsingular, ramified over the (reduced) curve $B \subseteq Y$, assumed to have at worst ADE singularities. Let $\mathcal{L}$ be an invertible sheaf on Y defined by $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$, which determines the double cover γ. Consider the following diagram*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\tau} & X \\ {\scriptstyle\overline{\gamma}}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \overline{Y} & \xrightarrow[\sigma]{} & Y \end{array}$$

where σ is a sequence of blow ups resolving the singularities of B, τ is a sequence of blow ups resolving X, the surface $\overline{X}$ is nonsingular, and $\overline{\gamma}$ is the double cover branched over B_{res}. Then the diagram commutes and

$$\omega_{\overline{X}} = (\gamma \circ \tau)^* \left(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{L} \right).$$

Furthermore, if L denotes the divisor associated to the line bundle $\mathcal{L}$ above, then the following equations hold:

(1) $\chi(\overline{X}) = 2\chi(Y) + \frac{1}{2}(L \cdot K_Y) + \frac{1}{2}(L \cdot L)$
(2) $p_g(\overline{X}) = p_g(Y) + h^0(Y, \mathcal{O}_Y(K_Y + L))$.
(3) $K^2_{\overline{X}} = 2K^2_Y + 4(L \cdot K_Y) + 2(L \cdot L)$.

Proof. Theorem 7.2 from Chapter III and Section 22 from Chapter V of [3]. □

The following is an example of Construction 2.17, which we will be fundamental in the proof of our main theorem.

Example 2.19. Let $Y = \mathbb{C}^2$ and let B be the divisor given by the equation $x^2 + y^{2m} = 0$. It follows that B is singular at the point

$$p_0 = (0, 0) \in \mathbb{C}^2.$$

Blow up Y at p_0 via $\sigma_1 : Y_1 \to Y$. Because B is given by $x^2 + y^{2m} = 0$ in local coordinates, $\mu_{p_0}(B) = 2$. By Proposition 2.5, $\sigma_1^* B = \overline{B} + 2D_1$, where

D_1 is the exceptional curve of the blow up σ_1. This means that $D_1^2 = -1$. By making the change of coordinates $(x, y) \mapsto (uv, v)$, on an affine open of Y_1 the pullback of B is given in local coordinates by $v^2(u^2 + v^{2m-2}) = 0$. In particular, $2D_1$ has local equation

$$v^2 = 0$$

and $\overline{B}$, the strict transform of B, has local equation

$$u^2 + v^{2m-2} = 0$$

By Construction 2.17, we set $B_1 = \overline{B}$ as $\mu_{p_0}(B) = 2$. Let $p_1 \in Y$ be the point $(0, 0)$ in the local coordinates u, v. Because B_1 is locally given by $u^2 + v^{2m-2} = 0$, it meets the point $p_1 = (0, 0)$ with multiplicity two. Furthermore, assuming $m > 1$, $p_1 \in B_1$ is singular, so we blow up at p_1 via $\sigma_2 : Y_2 \to Y_1$.

By Proposition 2.5, we have $\sigma_2^* B_1 = \overline{B}_1 + 2D_2$. Making the change of coordinates $(u, v) \mapsto (st, t)$ on an affine patch, the divisor $\sigma_2^* B_1$ is given in local coordinates s, t by

$$t^2 \left(s^2 + t^{2m-4} \right) = 0.$$

Because $\mu_{p_1}(B_1) = 2$, Construction 2.17 dictates we set $B_2 = \overline{B}_1$, given in local coordinates by

$$s^2 + t^{2m-4} = 0.$$

Furthermore, the exceptional curve D_2 has $D_2^2 = -1$ and the strict transform of D_1, now has self intersection -2 because we blew up at $p_1 \in D_1$. By abuse of notation, we denote the strict transform of D_1 by D_1.

Iterate this procedure until we obtain a branch divisor which is nonsingular. In particular, we blow up m times. This is because on the mth blow up, the pullback $\sigma_m^* B_{m-1} = \overline{B}_{m-1} + 2D_m$ is given in local coordinates by

$$v^2(u^2 + 1) = 0.$$

Because $\overline{B}_{m-1}$ is given by

$$u^2 + 1 = 0$$

and $2D_m$ by

$$v^2 = 0,$$

we set $B_m = \overline{B}_{m-1}$. Since in local coordinates B_m is nonsingular, the divisor B_m is nonsingular and we stop resolving. Let $\overline{Y} = Y_m$ and let

$$\sigma : \overline{Y} \to Y$$

denote $\sigma_m \circ \cdots \circ \sigma_1$.

The curves B_m and D_m meet transversally in two points. Additionally, notice that the exceptional curve D_m has self-intersection $D_m^2 = -1$ in contrast to the other curves in the exceptional divisor of σ, all of which have $D_i^2 = -2$. This is because after blowing up at the point p_{i-1} and creating D_i, we blew up the point $p_i \in D_i$, decreasing the self-intersection of D_i from -1 to -2, for $1 \leq i \leq m-1$. Furthermore, when $|i - j| = 1$, we have $D_i \cdot D_j = 1$, because one of these rational curves was created by blowing up at a point lying on the other. In summary, σ has produced a chain of rational curves, $D_1 \cup \cdots \cup D_m$, which satisfy

- $D_i^2 = -2$, for $1 \leq i \leq m-1$,
- $D_m^2 = -1$,
- $D_i \cdot D_j = 1$ for $|i - j| = 1$, and
- $D_i \cdot D_j = 0$ for $|i - j| > 1$.

Additionally, $B_{res} = B_m$ (the resolved branch divisor of B) meets D_m transversally in two points and is disjoint from all the other D_i.

Let $\overline{\gamma} : \overline{X} \to \overline{Y}$ be the double cover branched over B_{res}. For $i \neq m$, the rational curve D_i is disjoint from B_{res}, so

$$\overline{\gamma}^* D_i = E_i + E_i',$$

where $E_i, E_i' \simeq D_i$, and $E_i \cap E_i' = \varnothing$. As a consequence, for $i \neq m$, both E_i and E_i' are rational and $E_i^2 = (E_i')^2 = -2$, hence they are (-2)-curves. Now for $i = m$, define

$$E_m = \overline{\gamma}^* D_m.$$

Then E_m is a rational curve on $\overline{X}$ by the Hurwitz formula, and since $\overline{\gamma}$ is a double cover, $E_m^2 = -2$, which means that E_m is also a (-2)-curve. Furthermore, for $|i - j| = 1$ we have $E_i \cdot E_j = 1$ and $E_i' \cdot E_j' = 1$ and $E_m \cdot E_{m-1}' = 1$ by the relations among the D_i's. Thus, these curves assemble to yield the Hirzebruch-Jung string $[2, \ldots, 2]$ of length $2m - 1$.

3. Stable Surfaces with T-Singularities

3.1. **Main Results.** In this section, we prove that under mild hypotheses, there exists a stable surface, minimal and of general type which has a T-singularity of the form $\frac{1}{(k-2)^2}(1, k-3)$. In particular, the minimal resolution of this surface contains a Hirzebruch-Jung string of the form $[k, 2, \ldots, 2]$ of length $k - 3$.

Proposition 3.1 ([10]). *Let a and n be non-negative integers with $a \geq 1$ and $n \geq 2$. Let $Y = \mathbb{F}_n$ and suppose $B \sim 6h + 2af$ has at worst ADE singularities. Consider the commutative square from Proposition 2.18*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{\sigma}} & X \\ {\scriptstyle\overline{\gamma}}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \overline{Y} & \xrightarrow[\sigma]{} & Y = \mathbb{F}_n \end{array}$$

where γ is the double cover of Y branched over B, $\overline{\sigma}$ is the minimal resolution of X, the map σ is a sequence of blow ups resolving the singularities of B, and $\overline{\gamma}$ is the double cover branched over B_{res}, the resolved branch divisor of B. Then the surface $\overline{X}$ is minimal and of general type with the following invariants:

(1) $K_{\overline{X}}^2 = 6n + 4a - 8$;
(2) $p_g(\overline{X}) = 3n + 2a - 2$;
(3) $q(\overline{X}) = 0$; *and*
(4) $\chi(\overline{X}) = 3n + 2a - 1$.

Proof. We will show that the surface $\overline{X}$ is minimal and has the given invariants. We then use the Enriques-Kodaira classification theorem to conclude it is of general type.

Because $\mathbb{F}_n$ is minimal and γ is a double cover, the surface X must also be minimal. This is because if we did have some (-1)-curve on X, call it C, the pushforward $\gamma_* C$ would also have to be a (-1) curve. This cannot happen as $\mathbb{F}_n$ is minimal, so X must be minimal. Finally, since $\overline{X}$ is the minimal resolution of X, and X has no worse than ADE singularities, it follows that $\overline{X}$ must also be minimal.

Recall that $K_{\mathbb{F}_n} \sim -2h + (n-2)f$ and $L \sim \frac{1}{2}B \sim 3h + af$, where L is the divisor associated to the line bundle determining γ. Now, by Theorem 2.6, to show $\overline{X}$ is of general type, we only need to show $K^2_{\overline{X}} > 0$ and $p_g(\overline{X}) > 0$. By Theorem 2.18,

$$\begin{aligned} K^2_{\overline{X}} &= 2K^2_{\mathbb{F}_n} + 4(L \cdot K_{\mathbb{F}_n}) + 2(L \cdot L) \\ &= 16 + 4(-6n + 3(n-2) - 2a) + 2(9n + 6a) \\ &= 6n + 4a - 8. \end{aligned}$$

Because $n \geq 2$ and $a \geq 1$, it follows that $K^2_{\overline{X}} > 0$. Again by Theorem 2.18, we have

$$p_g(\overline{X}) = p_g(\mathbb{F}_n) + h^0(\mathbb{F}_n, \omega_{\mathbb{F}_n} \otimes \mathcal{L}) = h^0(\mathbb{F}_n, \omega_{\mathbb{F}_n} \otimes \mathcal{L}) \tag{3.2}$$

To compute this, we first show that L is ample, and then apply Riemann-Roch and Kodaira vanishing.

To see the ampleness of L, first observe that $L^2 = 9n + 6a > 0$. Now let C be an irreducible curve on $\mathbb{F}_n$. By Proposition 2.10, C is either the unique curve on $\mathbb{F}_n$ with $C^2 = -n$ or $C \sim \alpha h + \beta f$, where $\alpha, \beta \geq 0$ and $\alpha \neq 0$ or $\beta \neq 0$. In the former case,

$$L \cdot C = (3h + af) \cdot (h - nf) = 3n - 3n + a > 0.$$

In the latter,

$$L \cdot C = (3h + af) \cdot (\alpha h + \beta f) = (3n + a)\alpha + 3\beta > 0.$$

Since in either case $L \cdot C > 0$, the divisor L is ample by the Nakai-Moishezon criterion. Thus, Kodaira vanishing implies that

$$h^1(\mathcal{O}_{\mathbb{F}_n}(K_{\mathbb{F}_n} + L)) = h^2(\mathcal{O}_{\mathbb{F}_n}(K_{\mathbb{F}_n} + L)) = 0.$$

Furthermore, because $\mathbb{F}_n$ is rational, $\chi(\mathcal{O}_{\mathbb{F}_n}) = 1$. These two facts, along with Riemann-Roch, imply that

$$\begin{aligned} \chi(\mathcal{O}_{\mathbb{F}_n}(K_{\mathbb{F}_n} + L)) &= h^0(\mathcal{O}_{\mathbb{F}_n}(K_{\mathbb{F}_n} + L)) \\ &= \chi(\mathcal{O}_{\mathbb{F}_n}) + \frac{1}{2}(K_{\mathbb{F}_n} + L) \cdot L \\ &= 1 + \frac{1}{2}(h + (n + a - 2)f) \cdot (3h + af) \\ &= 1 + 3n + 2a - 3 \\ &= 3n + 2a - 2. \end{aligned}$$

Using equation (3.2) above, we have

$$p_g(\overline{X}) = 3n + 2a - 2,$$

because $p_g(\mathbb{F}_n) = 0$. Again by Theorem 2.18, since

$$\chi(\overline{X}) = 2\chi(\mathbb{F}_n) + \frac{1}{2}(L \cdot K_{\mathbb{F}_n}) + \frac{1}{2}(L \cdot L),$$

it follows that

$$\begin{aligned} \chi(\overline{X}) &= 2\chi(\mathbb{F}_n) + \frac{1}{2}(3h + af) \cdot (-2h + (n-2)f) + \frac{1}{2}(3h + af) \cdot (3h + af) \\ &= 2 + \frac{1}{2}(6n + 4a - 6) \\ &= 3n + 2a - 1. \end{aligned}$$

Because $\chi = 1 - q + p_g$, and

$$\chi(\overline{X}) = (3n + 2a - 2) + 1 = p_g(\overline{X}) + 1,$$

we conclude that $q(\overline{X}) = 0$.

Finally, because $n \geq 2$ and $a \geq 1$, we have $p_g(\overline{X}) > 0$ for all n and a. Thus, $\overline{X}$ is of general type by the Enriques-Kodaira classification theorem. □

Proposition 3.3. *Fix positive integers $k \geq 5$ and $n \geq 2$ so that $k - 2n \geq 1$. Let $B \subseteq \mathbb{F}_n$ be in the linear system $|6h + 2(k-2n)f|$. Suppose that B is given in local coordinates by the equation $x^2 + y^{2(n+k)} = 0$, meeting h with maximal tangency. That is, h is given by $y = 0$ in local coordinates. Then the surface $\overline{X}$ given by the minimal resolution of the double cover of $\mathbb{F}_n$ branched over B contains a Hirzebruch-Jung string $[k, 2, \ldots, 2]$ of length $k - 3$.*

Proof. Let $B \subseteq \mathbb{F}_n$ be in the linear system $|6h + 2(k-2n)f|$. Suppose that B is given in local coordinates by the equation $x^2 + y^{2(n+k)} = 0$, and set $m = n + k$. If we proceed as in Example 2.19, by repeatedly blowing up points of B which also lie on h, we obtain a map $\sigma : \overline{Y} \to Y$, whose exceptional divisor is a chain of rational curves $D_1, \ldots, D_m$. As in Example 2.19 B_{res}, the resolved branch divisor of B, meets the curve D_m transversally in two points. Moreover, B_{res} is disjoint from $\overline{h}$, the strict transform of h. This is because in blowing up repeatedly, we decrease the tangency of B to h until they no longer meet one another. Furthermore, since each point we have blown up lies on h or one of its strict transforms, and h is smooth, we have decreased the self-intersection of h by a total of $m = n + k$, i.e. $\overline{h}^2 = -k$. By defining $D_{m+1} = \overline{h}$, it is immediate that D_{m+1} is disjoint from B_{res} and that $D_{m+1}^2 = -k$.

In summary, for $1 \leq i, j \leq m + 1$,

$$D_i \cdot D_j = \begin{cases} 0, & |i-j| > 1 \\ 1, & |i-j| = 1 \\ \begin{cases} -2, & i = j \leq m-1 \\ -1, & i = j = m \\ -k, & i = j = m+1 \end{cases}, & |i-j| = 0 \end{cases}$$

Now, take the double cover $\overline{\gamma} : \overline{X} \to \overline{Y}$ with respect to B_{res}. Because $D_m \cdot B_{res} = 0$ for $i \neq m$, and $D_m \cdot B_{res} = 2$, we see that for $i \neq m$, the pullback of D_i is given by

$$\overline{\gamma}^* D_i = E_i + E_i'$$

where $D_i \simeq E_i$, $D_i \simeq E_i'$ and $E_i \cap E_i' = \varnothing$. Immediately, for all $i \neq m$ the curves E_i and E_i' are rational. Furthermore, because $(\overline{\gamma}^* D_i)^2 = 2(D_i)^2$ and $E_i \cap E_i' = \varnothing$, we conclude that for $i \neq m, m+1$, we have $(E_i)^2 = (E_i')^2 = -2$ and for $i = m+1$, we have $(E_{m+1})^2 = (E_{m+1}')^2 = -k$. Now for $i = m$, the pullback is given by

$$\overline{\gamma}^* D_m = E_m,$$

for E_m an irreducible curve on $\overline{X}$. Again, because $\overline{\gamma}$ is a double cover, $(E_m)^2 = (\overline{\gamma}^* D_m)^2 = -2$. By the Hurwitz formula, a curve over $\mathbb{P}^1$ branched at two points is genus zero curve, so it follows that $2g(E_m) = 0$. Thus, E_m is a (-2)-curve.

The intersection-theoretic properties of the curves D_i determine exactly how the curves on $\overline{X}$ meet. In particular,

$$E_i \cdot E_j = \begin{cases} 0, & |i-j| > 1 \\ 1, & |i-j| = 1 \end{cases}$$

and

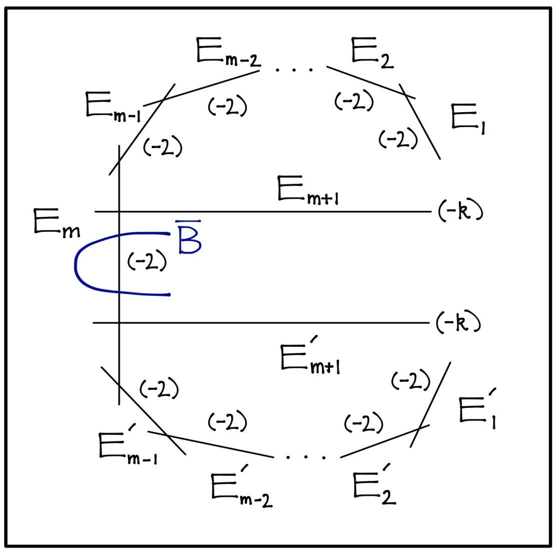
$$E'_i \cdot E'_j = \begin{cases} 0, & |i-j| > 1 \\ 1, & |i-j| = 1 \end{cases}$$

with

$$E_m \cdot E'_{m+1} = E_m \cdot E'_{m-1} = 1.$$

The relations between all the E_i and E'_i are summarized in Figure 2.

FIGURE 1. The Surface $\overline{X}$

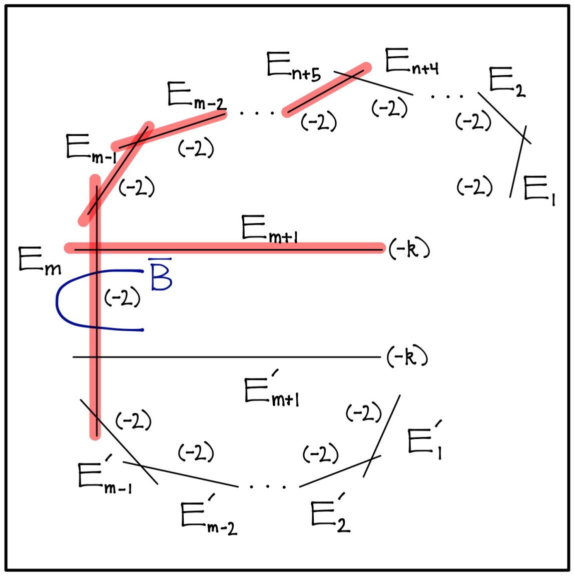


Let $H = E_{m+1} + E_m + \cdots + E_{n+5}$. By the relations above, for $n+5 \leq i, j \leq m+1$,

$$E_i \cdot E_j = \begin{cases} 1, & |i-j| = 1 \\ 0, & |i-j| > 1 \\ \begin{cases} -2, & i = j \neq m+1 \\ -k, & i = j = m+1 \end{cases}, & |i-j| = 0 \end{cases}$$

It follows that H is a Hirzebruch-Jung string of the form $[k, 2, \ldots, 2]$, and since $m+1-(n+5) = n+k+1-(n+5) = k-4$, the string is of length $k-3$. Figure 2 below illustrates the Hirzebruch-Jung string H on $\overline{X}$. $\square$

FIGURE 2. The Hirzebruch-Jung String H



We will use the surface $\overline{X}$ constructed in Proposition 3.3 to prove the existence of a minimal surface of general type with T-singularity $\frac{1}{(k-2)^2}(1, k-3)$. In particular, we will contract a Hirzebruch-Jung string on $\overline{X}$. The next two results with be used in the proof of Theorem 3.6.

Lemma 3.4. *Let Y be an algebraic surface with a T-singularity and let $\varphi : X \to Y$ be a resolution. If X is minimal and of general type, then Y is minimal and of general type.*

Proof. The resolution $\varphi : X \to Y$ is equivalently a contraction of the Hirzebruch-Jung string determining the T-singularity. Because φ is a contraction of finitely many rational curves, it is birational, so

$$p_g(Y) = p_g(X) > 0.$$

Furthermore, a contraction does not create any (-1)-curves, so Y is minimal as well. By Proposition 20 from [9],

$$K_Y^2 = K_X^2 + r,$$

where r is the length of the contracted Hirzebruch-Jung string. Thus,

$$K_Y^2 = K_X^2 + r > 0,$$

so by the Enriques-Kodaira classification theorem, Y is of general type. □

Proposition 3.5 ([3])**.** *Let X be a minimal surface of general type and C an irreducible curve on X. Then $K_X \cdot C \geq 0$ with equality exactly when C is a (-2)-curve.*

Proof. This is Corollary 2.3 from Chapter VII of [3]. □

Theorem 3.6. *Assume the hypotheses of Proposition 3.3. If we contract the divisors H, G_1 and G_2 (as defined below), as well as any other (-2)-curves on $\overline{X}$, we obtain a stable surface, W, minimal and of general type with T-singularity $\frac{1}{(k-2)^2}(1, k-3)$. Furthermore, W has invariants given by the following equations:*

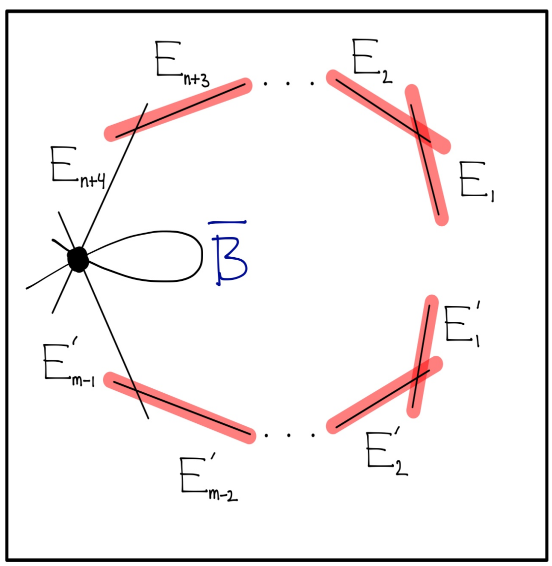
(1) $K_W^2 = 5k - 2n - 11$*;*
(2) $p_g(W) = 2k - n - 2$*;*
(3) $\chi(W) = 2k - n - 1$*; and*
(4) $q(W) = 0$*.*

Proof. Let $\varphi : \overline{X} \to W$ be the contraction of the divisors

$$\begin{aligned} H &= E_{m+1} + E_m + \cdots + E_{n+5} \\ G_1 &= E_{n+3} + \cdots + E_1, \text{ and} \\ G_2 &= E_1' + \cdots + E_{m-2}' \end{aligned}$$

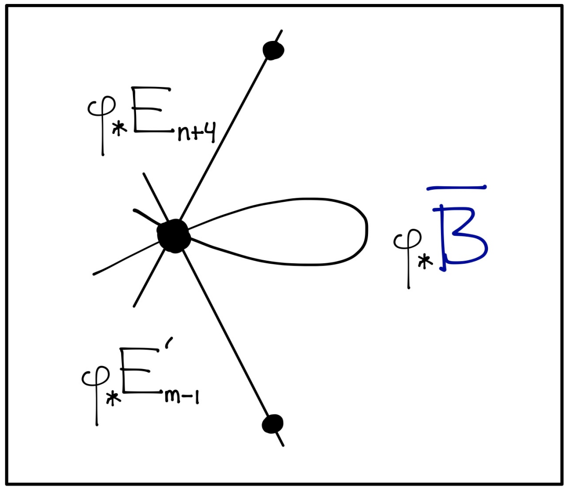
followed by the contraction of any remaining (-2)-curves on $\overline{X}$. We can view φ as first contracting H to obtain a surface V containing the chains G_1 and G_2. This is illustrated in Figure 3.

FIGURE 3. The Surface V



In contracting H, the images of E_{n+4} and E'_{m-1} are no longer necessarily (-2)-curves, as they meet the newly-formed T-singularity. Thus, we do not contract them. Then the chains G_1 and G_2 are contracted, along with any remaining (-2) curves. The resulting surface W has a T-singularity and multiple ADE singularities. This is illustrated in Figure 4 below.

FIGURE 4. The Surface W



Because ADE singularities do not affect the canonical class, by Lemma 3.4, W is minimal and of general type. Furthermore, by Proposition 2.16, the surface W has semi log canonical singularities. Thus, to show that W is stable, it remains to prove that K_W is ample.

By Proposition 3.5, for all irreducible curves $C \subseteq W$, the intersection product $K_W \cdot C \geq 0$ and $K_W \cdot C = 0$ if and only if C is a (-2)-curve. Since the only curves on W which are possibly (-2)-curves are $\varphi_* E_{n+4}$ and $\varphi_* E'_{m-1}$, it suffices to show that $K_W \cdot \varphi_* E_{n+4} > 0$ and $K_W \cdot \varphi_* E'_{m-1} > 0$. By the projection formula for intersection products,

$$K_W \cdot \varphi_* E_{n+4} = \varphi^* K_W \cdot E_{n+4}, \text{ and } K_W \cdot \varphi_* E'_{m-1} = \varphi^* K_W \cdot E'_{m-1}.$$

Because the T-singularity and ADE singularities on W are log canonical (see [11]) the pullback of K_W is given by

$$\varphi^* K_W = K_{\overline{X}} + \sum_{j=n+5}^{m+1} a_j E_j,$$

where $a_j > 0$ for all j. Computing the intersection products, we find that

$$\begin{aligned} \varphi^* K_W \cdot E_{n+4} &= \left(K_{\overline{X}} + \sum_{j=n+5}^{m+1} a_j E_j \right) \cdot E_{n+4} \\ &= a_{n+5} E_{n+5} \cdot E_{n+4} \\ &= a_{n+5} > 0 \end{aligned}$$

and

$$\begin{aligned}\varphi^* K_W \cdot E'_{m-1} &= \left(K_{\overline{X}} + \sum_{j=n+5}^{m+1} a_j E_j\right) \cdot E'_{m-1} \\ &= a_m E_m \cdot E'_{m-1} \\ &= a_m > 0.\end{aligned}$$

These computations combined with the fact that $K_W^2 = K_{\overline{X}}^2 + k - 3 > 0$ (Lemma 3.4), imply, by the Nakai-Moishezon criterion, that K_W is ample. Thus, the surface W is stable, minimal, and of general type.

The verification of formulae (2)-(4) follow readily from the formulae in Proposition 3.1 by substituting $k - 2n$ for a. The formula (1) follows in virtue of the fact that $K_W^2 = K_{\overline{X}}^2 + r$ (Proposition 20 in [9]), by substituting $a = k - 2n$ and $r = k - 3$. □

3.2. **Consequences and Examples.** Throughout these examples, assume that we can choose $B \sim 6h + 2(k - 2n)f$ on $\mathbb{F}_n$ so that it satisfies the hypotheses of Theorem 3.6.

Example 3.7. Fix $k = 6$, so that surface we construct has a $\frac{1}{16}(1,3)$ singularity. To have $k - 2n \geq 1$ the equality $n = 2$ is forced. Furthermore, since $k = 6$, it follows that $r = k - 3 = 3$. By the formulae given in Theorem 3.6, we have

- $K_W^2 = 6(2) + 4(2) - 8 + 3 = 12 + 3 = 15$;
- $p_g(W) = 3(2) + 2(2) - 2 = 8$; and
- $\chi(W) = 3(2) + 2(2) - 1 = 9$.

This implies that for $k = 6$, we obtain a surface W which lies in the moduli space $\overline{M}_{15,9}$, sitting over the Noether line by 3.

Example 3.8. Fix $k = 7$, so that the surface we construct has a $\frac{1}{25}(1,4)$ singularity. Notice that $n = 2$ and $n = 3$ satisfy $k - 2n \geq 1$, so we consider both. When $n = 2$, we have

- $K_W^2 = 6(2) + 4(3) - 8 + 4 = 20$;
- $p_g(W) = 3(2) + 2(3) - 2 = 10$; and
- $\chi(W) = 3(2) + 2(3) - 1 = 11$.

Thus, W lies in $\overline{M}_{20,11}$ and sits over the Noether line by 4. When $n = 3$, we have

- $K_W^2 = 6(3) + 4(1) - 8 + 4 = 18$;
- $p_g(W) = 3(3) + 2(1) - 2 = 9$; and
- $\chi(W) = 3(3) + 2(1) - 1 = 10$.

Thus, W lies in $\overline{M}_{18,10}$, and sits over the Noether line by 4.

Theorem 3.6 provides one way to construct stable surfaces with the specified invariants. Another such way is provided by Persson in [10], via genus two fibrations. In particular, it follows from Theorem 3.9 that there are smooth minimal surfaces of general type with the invariants in Proposition 3.3 which are genus two fibrations.

Theorem 3.9. *Let x and y be positive integers satisfying*

$$2x - 6 \leq y \leq 8x \text{ and } y \neq 8x - m$$

(where $m = 2$, or m is odd and $1 \leq m \leq 15$ or $m = 19$). Then there exists a minimal surface of general type X with $K_X^2 = y$ and $\chi(X) = x$ which is a genus two fibration.

Proof. This is one of the main results in [10]. □

Corollary 3.10. *The invariants of W (as constructed in Theorem 3.6) satisfy the above inequality, so there exists a genus two fibration X with $K_X^2 = K_W^2$ and $\chi(X) = \chi(W)$.*

Proof. By the formulae given in Proposition 3.1, we have $2\chi - 6 = 2(3n + 2a - 1) - 6 = 6n + 4a - 8 = K_X^2$ and $8\chi(W) = 24n + 16a - 8$. As $K_X^2 \le K_W^2$ we only need check $K_W^2 \le 8\chi$. Indeed, it is certainly the case that

$$6n + 4a - 8 + r \le 24n + 16a - 8,$$

as n and a are positive and nonzero integers. Furthermore, $8\chi(W) - K_W^2 = 24n + 16a - 8 - (6n + 4a - 8 + r) = 18n + 12a - r$, so for $a = k - 2n \ge 1$ and $r = k - 3$, we have

$$\begin{aligned} 8\chi(W) - K_W^2 &= 18n + 12(k-2n) - (k-3) \\ &= 11k - 6n + 3 \\ &\ge 11(2n+1) - 6n + 3 \\ &= 16n + 14 \\ &\ge 46. \end{aligned}$$

This inequality guarantees that none of the exceptional cases of Theorem 3.9 hold. In particular, $K_W^2 - 8\chi(W) \ge 46$ implies that $K_W^2 \ne 8\chi(W) - m$ for $m = 2$ or m odd and $1 \le m \le 15$ or $m = 19$. Thus, by Theorem 3.9 the claim follows. □

4. Questions and Future Work

Question 4.1. *Is there a way to remove the hypothesis in Proposition 3.3 and Theorem 3.6 that B must be given in local coordinates by $x^2 + y^{2(n+k)} = 0$?*

In order to determine whether or not this can be done, we would have to compute $h^0(\mathcal{O}_{\mathbb{F}_n}(6h + 2(k-2n)f))$, and argue that the dimension of $|6h + 2(k-2n)f|$ gives us sufficiently many degrees of freedom to choose $D \in |6h + 2(k-2n)f|$ given in the local coordinates by the equation above. Unfortunately, this is a complicated problem, and the author did not have enough time to determine whether the technical hypothesis can be lifted.

Question 4.2. *Could we use Persson's work in* [10] *to construct the surfaces we have built in Theorem 3.6? Theorem 3.9 provides a way to construct surfaces with invariants equal to those from Theorem 3.6, but it is unknown whether these surfaces have the T-singularities* $[k, 2, \ldots, 2]$.

The author needs to further familiarize himself with the work in [10] in order to make assertions about next steps for Question 4.2.

Question 4.3. *Let $k \ge 5$. Let W be the surface constructed in Theorem 3.6 with a $\frac{1}{(k-2)^2}(1, k-3)$ singularity, and let $\overline{M}_{K^2,\chi}(k)$ be the moduli space in which W lies. Now if we consider a $\mathbb{Q}$-Gorenstein smoothing of W, that is, a one-parameter $\mathbb{Q}$-Gorenstein deformation with smooth general fiber, this smoothing determines a divisor in the boundary of $\overline{M}_{K^2,\chi}(k)$. Does this divisor ever determine an embedding into projective space? If so, when does it, and if not, why?*

Initial steps towards answering this question would be to explicitly construct $\mathbb{Q}$-Gorenstein smoothings of W for small k. Hopefully these examples would inform us about the general case.

5. Acknowledgments and Dedication

First and foremost, I would like to thank my thesis advisor Professor Julie Rana. Her guidance and expertise have been integral to the completion of this thesis. Furthermore, her encouragement, support, and friendship have helped me persevere through the trials of research and writing.

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I would like to dedicate this thesis to my parents, Patricia and Patrick, and to my partner, Emma. Mom and Dad, your unconditional love and support of my interests means more to me than I can put into words. I am so thankful to have such wonderful parents and role models. Emma, you bring joy into my life every day and your patience knows no bounds. Without your love and support, I could not have finished this thesis.

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