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Development of Utility Theory and Utility Paradoxes

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ABSTRACT:[1]

Since the pioneering work of von Neumann and Morgenstern in 1944[2] there have been many developments in Expected Utility theory. In order to explain decision making behavior economists have created increasingly broad and complex models of utility theory. This paper seeks to describe various utility models, how they model choices among ambiguous and lottery-type situations, and how they respond to the Ellsberg and Allais paradoxes. This paper also attempts to communicate the historical development of utility models and provide a fresh perspective on the development of utility models.

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[1] This paper would not have been possible without the guidance provided by Professor Adam Galambos (Lawrence University, Appleton Wi.), and Professor Merton Finkler (Lawrence University, Appleton, Wi.), to whom I am extremely grateful.
[2] Neumann, John von, Morgenstern, Oskar, *Theory of Games and Economic Behavior*. Princeton N.J., U.S.A: Princeton University Press, 1944

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Section 1: The Differences Between Utility Theorists

1.1: Introduction

There are many economists who focus on Expected Utility, Behavioral Economics, Bounded-Rationality theory, and other branches that have direct connections to utility theory. Two major pioneers of utility theory are John von Neumann and Oskar Morgenstern with their axiomatization of the Expected Utility model in 1944. Soon afterwards, paradoxes arose that showed that the von Neumann-Morgenstern Expected Utility model was not applicable in every lottery-type situation. Several economists, namely, Maurice Allais, Daniel Ellsberg, and Mark Machina, identified several paradoxes to various utility theories. These paradoxes have shown that economists need better utility models, and with that necessity, came various Non-Expected Utility models such as Rank-Dependent Utility, Cumulative Prospect Theory, Maxmin Expected Utility theory, amongst many others. These models have shown the trends in utility theory and their development either empirically or abstractly. Despite many obstacles, it seems that economists have finally discovered solutions to these paradoxes, though there have been many barriers. These impediments have paved the way to an important discovery, which is that the current utility models are suitable for modeling utility and predicting decisions, in particular settings, but they are not always applicable. Dani Rodrik (2015) explains a similar phenomenon in his book *Economics Rules: The Rights and Wrongs of the Dismal Science*, though in the context of macroeconomics.[3] To fully understand the Non-Expected Utility models and the responses given to each model, it is necessary to start by turning to the recent literature on Bounded-Rationality theory and Neoclassical utility theory.

This paper uses the development of utility theory, as well as the paradoxes that are associated with utility theories, and argues that economists should be viewing the history of thought within utility theory differently. In order to develop that argument, I take several

[3] Rodrik, Dani, *Economics Rules: The Rights and Wrongs of the Dismal Science.* New York, New York, U.S.A: W. W. Norton & Company, Inc., 2015

different utility models and discuss their theorems and axioms in order to show the differences in these various models and display how each model independently resolves several paradoxes to the von Neumann-Morgenstern Expected Utility model. By showing the differences in each model and its resolutions to the paradoxes, I am able to develop the argument that different utility models, developed through various theoretical approaches ranging from neoclassical to behavioral, are related. I also discuss the neoclassical view of utility theory and show that all of these models, despite the differences in economic modeling assumptions, are connected, and relevant for different situations of choice under ambiguity. I take those differences in development and create a utility theory development tree to explicitly display each model with its connections to different research branches in economics. I do this to contrast the traditional view of the linear development of utility theories which is shown when I discuss the trends of utility models.

1.2: Various Economic Thought Processes and Modeling

Matthew Rabin (2013)[4] lays out the ways in which neoclassical economists can use Bounded-Rationality, and limits on information in optimization, to properly model the behavior of agents. This suggests that the way to make progress in the study of Bounded-Rationality and utility models is to build on the Neoclassical Optimization model. Instead of discrediting neoclassicism, it is more beneficial for utility theorists to look at the successes of Neoclassical utility models and the successes of other branches of utility theories. Furthermore, in another paper, "*An interpretive history of challenges to neoclassical microeconomics and how they have fared*," Roberto Mazzoleni and Richard Nelson (2013)[5] are skeptical of the neoclassical views of economics. They are concerned with the increased amount of optimization and advanced mathematics used in economics. Their views in this paper claim that nonmathematical work is

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[4] Rabin, Matthew. "Incorporating Limited Rationality into Economics." *Journal of Economic Literature 2013*. Vol. 51(2): 528-543

[5] Mazzoleni, Roberto, Nelson, Richard. "An interpretive history of challenges to neoclassical microeconomics and how they have fared." *Journal of Industrial and Corporate Change*. Vol. 22(6): 1409-1451

being pushed out of economic publications. Sidney Winter (2013)[6] furthers the points made by Mazzoleni and Nelson, by stating that equilibrium, in the view of economists, is too frequently viewed only in the neoclassical sense. Winter broadens this view with the comment that the neoclassical commitment to optimizing actors has a high opportunity cost because it distorts the priorities of economics.

Winter positively responded to Mazzoleni and Nelson, and he attempted to challenge neoclassical models by stating that**,** "increasing mathematization of economic theorizing since World War II has amplified the negative consequences of the underlying flaws in the neoclassical account of economic life." This statement by Winter is rather strong, even for a modern economist, given the progress made in economics since the end of World War II. Winter then posed questions regarding the philosophy of economics: "Is economics a science, or could it be? What is it really about? On what basis should theoretical premises be assessed?" These are interesting questions that need to be addressed by using economic theory and through looking at the development of various models in economic theory. It is possible to answer Winter's question because the theoretical premise that economics bases itself on is the development of models. In the realm of utility theory, I seek to address the development of utility theory and how economists have generated better models that bring economics and the development of utility theory into a more scientific realm. However, it is necessary to view different branches of economic thought within utility theory so that cross examination and progress within utility theory can be achieved. By these different branches, I mean the different ways in which utility theory has developed, and that is though Behavioral [Economic] development, Neoclassical development, and Subjective (probability)[7] development. Before discussing the newer models and developments in utility theory, it is necessary to show how

[6] Winter, Sidney. "Optimization as a constraint: a comment on Mazzoleni and Nelson." *Journal of Industrial and Corporate Change*. Vol. 23(2): 613-631
[7] Where Subjective probability refers to probabilities derived from an agents' personal judgment of the outcomes of a lottery

utility theory came to the development of Non-Expected Utility models. This starts with the St. Petersburg Paradox, which eventually led to many interesting developments, most importantly, the axiomatization and model built by von Neumann and Morgenstern, which led to various paradoxes that are described in the next section.

Section 2: von Neumann-Morgenstern Expected Utility and the Paradoxes

2.1: St. Petersburg Paradox

Before explaining Non-Expected Utility theory, it is important to understand the origins of utility theory which occurred through the developments of the St. Petersburg, Allais, and Ellsberg paradoxes. Expected Utility theory was first proposed as a solution to the St. Petersburg paradox. The St. Petersburg paradox,[8] which challenges the expected value theory of utility, states:

Assume a casino offers a game of chance to players where a fair coin is tossed at each stage, and the pot starts at two dollars and doubles for each head. When the first tails appears, the game ends and the player collects whatever is in the pot. Therefore, the player wins 2^n dollars where n is the number of heads that have appeared plus one. So the question for the casino is, how much should a player be charged to play the game, or how much would a player be willing to pay in order to play this game? In order to come to an answer, early economists looked at the expected value of this game:

$$EV = \frac{1}{2} * 2 + \frac{1}{4} * 4 + \frac{1}{8} * 8 + \cdots$$

$$EV = 1 + 1 + 1 + \cdots$$

$$\therefore : EV = \infty$$

Clearly, people are not going to pay an infinite amount to play this game, and their expected payout will not be infinite, thus it became a paradox.

2.2: Expected Utility and Axioms

Expected Utility theory arose as a solution to the St. Petersburg paradox. The solution was proposed by Daniel Bernoulli, and was formalized by von Neumann and Morgenstern:[9]

$$\sum_{i=1}^{n} (p_i) U(\bar{x}_i) \tag{1}$$

[8] Mas-Colell A., Whinston M., and Green J., *Microeconomic Theory*, 6: "Choice Under Uncertainty." (Oxford University Press, Inc. 1995), 185.

[9] Mas-Colell A., Whinston M., and Green J., *Microeconomic Theory*, 6: "Choice Under Uncertainty." (Oxford University Press, Inc. 1995), 173

Here n is the number of states, U is the utility function of outcomes $\bar{x}$, and the p_i's are the probabilities associated with the outcomes. Bernoulli posited that people maximize their expected utility from a gamble, as opposed to the expected value of the gamble, which led to the above formula.

Four axioms for rationality within utility functions were developed to further formalize Expected Utility theory.[10] This axiomatization allowed economists to explain how expected utility maximizers act. Some necessary notation: a preference relation is denoted by $\succcurlyeq$ (which is a binary relation on the set of alternatives, $X \subset \mathcal{R}^n$, that allows preference comparisons between outcomes) where $x, y \in X \; and \; x \succcurlyeq y$ is read as "x is at least as good as y." Similarly, $x \succ y$ is read as "x is strictly preferred to y." And, $x \sim y$ is read as "x is indifferent to y."[11]

1. Completeness: $\forall x, y \in X, we\ have\ x \succcurlyeq y\ or\ y \succcurlyeq x\ or\ both\ (both\ implies\ x \sim y)$
2. Transitivity: $\forall x, y, z \in X\ if\ x \succcurlyeq y\ and\ y \succcurlyeq z, then\ x \succcurlyeq z$
3. Independence: $The\ preference\ relation\ \succcurlyeq$

   $on\ the\ space\ of\ simple\ lotteries,$[12] $\Lambda, satisfies\ the\ independence\ axiom\ if\ \forall\ \lambda, \lambda', \lambda'' \in$

   Λ and $\alpha \in (0,1)$, we have:

   $\lambda \succcurlyeq \lambda'\ iff\ \alpha\lambda + (1-\alpha)\lambda'' \succcurlyeq \alpha\lambda' + (1-\alpha)\lambda''$
4. Continuous: $The\ preference\ relation \succcurlyeq$

   $on\ the\ space\ of\ simple\ lotteries\ \ is\ continuous\ if\ for\ any\ \lambda, \lambda', \lambda'' \in \Lambda, the\ sets$:

   $\{\alpha \in [1,0]: \alpha\lambda + (1-\alpha)\lambda' \succcurlyeq \lambda''\} \subset [0,1]$

   And

[10] Mas-Colell A., Whinston M., and Green J., *Microeconomic Theory*, 3: "Classical Demand Theory." 6: "Choice Under Uncertainty." (Oxford University Press, Inc. 1995), 42, 171-172
[11] Mas-Colell A., Whinston M., and Green J., *Microeconomic Theory*, 3: "Classical Demand Theory." 1: "Preference and Choice." (Oxford University Press, Inc. 1995), 6
[12] Simple lotteries: $\lambda\ is\ a\ list\ of\ probabilities\ associated\ with\ outcomes, X,$
$\lambda = (p_1, \dots, p_N)\ with\ p_N \geq 0\ \forall\ n, and\ \Sigma_n p_n = 1, \qquad And\ there\ is\ some\ set\ of\ outcomes,$
$X = (x_1, \dots, x_n),\ each\ of\ which\ occurs\ with\ some\ known\ probability, p_i \in \lambda.$* (3)
* Mas-Colell A., Whinston M., and Green J., *Microeconomic Theory*, 3: "Classical Demand Theory." 6: "Choice Under Uncertainty." (Oxford University Press, Inc. 1995), 168

$\{\alpha \in [1,0]: \lambda'' \succcurlyeq \alpha\lambda + (1-\alpha)\lambda'\} \subset [0,1]$

are closed.[13] (2, a-d)

Another important aspect of this axiomatization was that it formalized basic preference theory in a way that had previously not been done. Daniel Bernoulli may have formalized the Expected Utility function, but the axiomatization allowed economists to relate the theory to choice behavior. These axioms led to many new models and a basis for utility theory. The Independence axiom is the most important to fully understand. It says that if we mix each of two lotteries with a third one, then the preference ordering of the two resulting mixtures does not depend on the third lottery used, hence independence.[14]

2.3: The Allais Paradox

Though the development of the von Neumann-Morgenstern Expected Utility model was, and still is, massively important, it led to the Allais (1953) and Ellsberg (1961) paradoxes. These paradoxes focus primarily on the independence axiom and show that agents cannot always be modeled using Expected Utility for choices under uncertainty. The Allais paradox[15] states:

Consider a lottery with three possible monetary prizes $(N = 3)$:

Prize A: 2,500,000 *Prize B*: 500,000 *Prize C*: 0

And, the agent chooses between lotteries with respective probabilities:

$\lambda_1 = (0,1,0)$ *and* $\lambda_1' = (.1, .89, .01)$ and $\lambda_2 = (0, .11, .89)$ *and* $\lambda_2' = (.1, 0, .9)$

Allais found that the common choices for agents in experimental situations:

$\lambda_1 \succ \lambda_1'$ *and* $\lambda_2' \succ \lambda_2$

The issue is that in the Expected Utility model, the decisions are not consistent.

[13] Closed sets: *Fix a set* $\Omega \subset \mathcal{R}^n$. *A set* $B \subset \Omega$ *is closed iff for every sequence* $\omega^m \to \omega \in \Omega$, *with* $\omega^m \in B\ \forall\ m$, *we have* $\omega \in B$ (4)

[14] Mas-Colell A., Whinston M., and Green J., *Microeconomic Theory*, 3: "Classical Demand Theory." 6: "Choice Under Uncertainty." (Oxford University Press, Inc. 1995), 171

[15] Allais, Maurice, "Le Comportement de l'Homme Ratinel devant le Risque, Critique des Postulates et Axiom de l'Ecole Americaine." *Econometrica*, October 1953, 503-546

To elaborate, suppose a von Neumann-Morgenstern Expected Utility function denoted by the values of the respective prizes: $u_{25},\ u_{05}, and\ u_0$. So the choice $\lambda_1 \succ \lambda_1'$ then implies:

$u_{05} > (.1)u_{25} + (.89)u_{05} + (.01)u_0$

$Adding: (.89)u_0 - (.89)u_{05}\ to\ each\ side:$

$(.11)u_{05} + (.89)u_0 > (.1)u_{25} + (.9)u_0;\ this\ yields:$

$55{,}000 \nsucc 250{,}000$

Therefore, preferences should state: $\lambda_2 \succ \lambda_2'$, which is not what experimental subjects generally choose. This displays an inconsistency in von Neumann-Morgenstern Expected Utility, because it shows the phenomenon of preference reversals, which is inconsistent with the independence axiom.

2.4: The Ellsberg Paradox

The next paradox to von Neumann-Morgenstern Expected Utility is the Ellsberg (1961)[16] paradox. This paradox is especially important for understanding Non-Expected Utility theory because it was the primary foundation that inspired Non-Expected Utility theories. The Ellsberg Paradox is as follows:

$Consider\ a\ single\ urn\ containing\ 30\ red\ balls\ and\ 60\ other\ balls\ that\ are\ either\ black\ or\ yellow, and\ the\ agent\ does\ not\ know\ how\ many\ black\ or\ yellow\ balls\ are\ in\ the\ urns,\ only\ that\ there\ are\ a\ total\ of\ 60\ black\ and\ yellow\ balls.\ An\ agent\ is\ given\ a\ choice\ of\ gambles\ \beta\ and\ \ \zeta\ in\ the\ first\ draw, and\ gambles\ \gamma\ and\ \delta\ in\ the\ next.$

$Where\ the\ outcomes\ are:$

$Draw\ 1:\ \ \beta: \$100\ from\ a\ red\ ball$ or $\zeta: \$100\ from\ a\ black\ ball$

$Draw\ 2: \gamma: \$100\ from\ a\ red\ or\ yellow\ ball$ or $\delta: \$100\ from\ a\ black\ or\ yellow\ ball$

The decision relies on uncertainty and probability, and the outcome is simply whether the ball is red or non-red: $\frac{1}{3}$ *v.* $\frac{2}{3}$ in the experimental setting. The experimental evidence appears to be

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[16] Ellsberg, Daniel. "Risk Ambiguity and the Savage Axioms." *Quarterly Journal of Economics*. Nov. 1961, Vol. 75, 643-669

problematic because von Neumann-Morgenstern Expected Utility would propose that in choosing between the two gambles, people account for the probability that non-red balls are yellow versus black, and then compute the expected utility from each gamble.

Subjects generally presented preferences $\beta \succ \zeta$ in draw one and, and $\delta \succ \gamma$ in draw 2. This means that agents uniformly believe that that there are more red than black balls in gamble one and similarly they believe that there are more black and yellow balls than red and yellow balls. Mathematically, this leads to a contradiction, by denoting the colors of the balls as R[ed], Y[ellow], and B[lack], and plugging the given probabilities into the von Neumann-Morgenstern Expected Utility function with preferences assumed to be: $U(\$100) \succ U(\$0)$: Agents usually choose:

Draw 1: $R[U(\$100) - U(\$0)] \succ B[U(\$100) - U(\$0)]\ therefore, R \succ B$

Draw 2: $B[U(\$100) - U(\$0)] \succ R[U(\$100) - U(\$0)]\ therefore, B \succ R$

For this to be true it would mean that in draw 1 agents believe that the probability of drawing a red ball, one third, is greater than drawing a black ball. For that to be true, it means that agents must think that there are more yellow balls than black balls such that the probability of drawing a black ball is less than one-third. However, in draw two, for agents to make this choice it means they believe that there are more in total of yellow plus black, as opposed to yellow plus red. But this does not make sense given that in draw one they believed that there were more red balls than black since there were more yellow than black balls. Represented mathematically:

Agents in draw one believe:

$p(R) > p(B)\ so\ they\ must\ believe\ that\ p(Y) > p(B)\ in\ the\ urn, so\ they\ believe\ R > B$

But in draw two:

$p(Y, B) > p(Y, R), so\ they\ believe\ B > R$

This presents a contradiction and it shows that agents are averse to ambiguity, because even though in the first case they believe that there were more yellow than black balls, thus more red than black; in draw two they believe that there were more yellow and black than red and black, thus more black than red balls in the urn.

Yet, these are not the only paradoxes to Expected Utility theory. There is also Machina's Paradox[17] and a variety of others that arise in lottery-like situations. Through this, there were major developments in the field of utility theory because it seemed obvious that the von Neumann and Morgenstern model was not fit to handle some types of ambiguous and lottery-type situations. However, the von Neumann and Morgenstern model brought much clarity to economics because through the axiomatizations Non-Expected Utility theorists were able to have a framework to build new models. These new models, which arose from these paradoxes and the original von Neumann and Morgernstern model, are each important in a variety of situations.

[17] Machina, Mark., "Choice Under Uncertainty: Problems Solved and Unsolved." *Economic Perspectives*. Summer 1987, Vol. 1, 121-154

Section 3: Non-Expected Utility Theories

3.1 Cumulative Prospect Theory

Due to these paradoxes there has been the birth and growth of the field of Non-Expected Utility theory. Major models in Non-Expected Utility theory include (Cumulative) Prospect Theory (1979,[18] 1992[19]), Maxmin Expected Utility (1989),[20] Rank-Dependent Utility theory (1982),[21] and Yaari's Dual Theory (1987)[22] (discussed in section 4). The first major model to be derived due to these paradoxes was Prospect Theory, which was published by Daniel Kahneman and Amos Tversky (1979). This model describes a two-stage decision process:

- In stage one, agents perceive various outcomes as gains and losses, as opposed to the von Neumann and Morgenstern model where agents maximize based on their belief of the final outcome. The outcomes are then judged as gains and losses based on some reference point.[23]
- Then in the following evaluation phase, agents "compute" a value function with outcomes and probabilities and then choose the alternatives with higher utility based on their objective functions.
- The most general form of the model:

  $$\Sigma_{i=1}^{n} v(x_i) * \pi(p_i) \qquad (5)$$

- This states that *v* is the expected utility for outcomes x_i, multiplied by the respective probabilities of each event p_i with decision weight π, where a decision weight is defined

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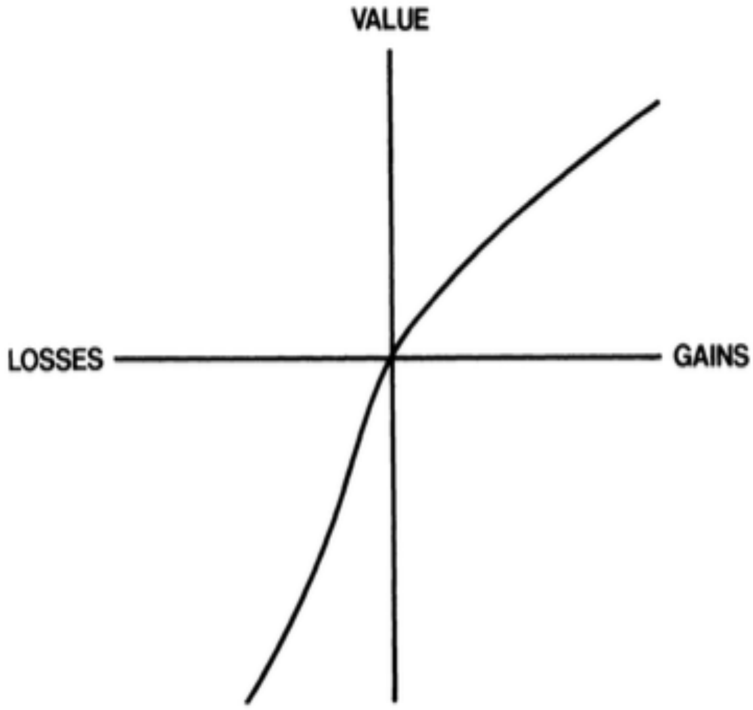
[18] Kahneman, Daniel, Tversky, Amos., "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*. March 1979, Vol. 47(2), 263-291
[19] Kahneman, Daniel., Tversky, Amos. "Advances in Prospect Theory: Cumulative Representation of Uncertainty." *Journal of Risk and Uncertainty*, 1992. Vol. 5, 297-323
[20] Gilboa, Itzhak, Schmeidler, David. "Maxmin Expected Utility with Non-Unique Prior." *Journal of Mathematical Economics*, 1989, Vol. 18, 141-153
[21] Quiggin, John. "A theory of anticipated utility." *Journal of economics behavior and organization*, Vol. 3(4), 323-343
[22] Yaari, Menahem. "The Dual Theory of Choice under Risk." *Econometrica*, Jan. 1987, Vol. 55(1), 95-115.
[23] Also see: Samuelson, Paul., "A Note on the Pure Theory of Consumer's Behavior." *Economica*, Feb. 1938., Vol. 5(17), 61-71

as some weight reflecting the impact the probability of an outcome has on an agents' decisions. Additionally, the value function passes through the reference point. The graph generally appears as:



[24]

This graph shows how people perceive gains and losses with the reference point being the origin. Prospect theory proposed that agents maximize total utility based on a reference point, as opposed to ignoring the reference point such that they maximize total utility for one set of outcomes, as it is assumed in the von Neumann-Morgenstern model. However, the original paper from 1979 was found to have a major problem, which was later solved by John Quiggin (1982). That problem was that the model violated first-order stochastic dominance, meaning that the following:

$Let\ \ \tilde{x}_a\ and\ \tilde{x}_b\ be\ random\ variables\ with\ cumulative\ distribution\ functions$

[24] Tversky, Amos, Kahneman, Daniel. "Rational Choice and the Framing of Decisions." *The Journal of Business*, Oct. 1986, Vol. 59(4-2), S251-S278

$F_a(x)$ and: $F_b(x)$. Then F_a first − order stochastically dominated F_b if $F_a(x) \leq F_b(x)$[25]

(6)

Essentially, since Prospect Theory occasionally violated first-order stochastic dominance, it is possible that a decision maker that would be described by Prospect Theory would prefer a lottery to one that stochastically dominates it.

Cumulative Prospect Theory resolved this issue and added various new axioms which made it different from Expected Utility theory and Prospect Theory. In Cumulative Prospect Theory, Kahneman and Tversky keep the axioms of completeness and transitivity, but change the independence axiom. They modify independence to Comonotonic Independence. In order to properly understand Comonotonic Independence, it is necessary to define strict monotonicity and comonotonicity with regards to preference relationships:

$$Strict\ Monotonicity: A\ Preference\ Relation \succcurlyeq on\ X\ is\ monotonic\ if\ \forall\ x, y \in X\ and\ y \gg x\ implies\ y \succ x.\ The\ Preference\ Relation\ is\ strictly\ monotonic\ if\ y \geq x\ and\ y \neq x\ which\ implies\ that\ y \succ x \qquad (7)$$

$$Comontonicity: Prospects^{26}\ f\ and\ g\ are\ comonotonic\ \ if\ there\ is\ are\ no\ pairs\ of\ states, s, s'\ such\ that: f(s) \succ f(s')\ and\ g(s) \preccurlyeq g(s') \qquad (8)$$

Since Kahneman and Tversky defined comonotonicity within prospects they then could define a new independence axiom, namely Comonotonic Independence, which is independence defined for comonotonic, strictly monotonic prospects. Comonotonic Independence:

$$Comontonic\ Independence\ requires\ that\ f \succ g \Rightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h\ \forall \alpha \in (0,1)\ and\ all\ f, g, h\ that\ are\ pairwise\ comonotonic \qquad (9)$$

$Where\ pairwise\ comonotonic\ is\ defined\ as\ comparisons\ between\ preference$

[25] Joshi, Mark, Paterson, Jane., *Introduction to Mathematical Portfolio Theory*. Cambridge Mass. U.S.A. Cambridge University Press. 134, 2013

[26] Defined on next page

$relations\ that\ are\ comonotonic$

This axiom says that Comonotonic Independence axiom holds for comonotonic prospects, but it does not require independence to hold for prospects that are not comonotonic.[27]

This is important because it implies a sense of ordering within the model, and that was a primary achievement from Quiggin in his Rank-Dependent Utility model. This method used by Quiggin helped Kahneman and Tversky to change their original Prospect Theory model to Cumulative Prospect Theory.[28] Additionally, Cumulative Prospect Theory also satisfies another condition, which is referred to as double matching:

$$\forall\ Prospects\ f, g \in F, if\ f^+ \approx g^+\ and\ f^- \approx g^-, then\ f \approx g,$$
$$where\ f^+ and\ g^+ \in \mathcal{R}^+\ and\ f^-\ and\ g^- \in \mathcal{R}^- \quad (10)$$

Whereas prospects (also known as acts) are defined as:

$$F = \{f: S \to X\}, where\ F\ is\ the\ set\ of\ all\ uncertain\ prospects, f.\ \ S\ is\ a\ finite\ set\ of\ states\ of\ nature, and\ X\ is\ the\ set\ of\ all\ possible\ outcomes. \quad (11)$$

The double matching postulate means that if the positive parts of prospects *f* and *g* and the negative portions of the prospects are indifferent, then the agent is indifferent to the prospects. From double matching and the use of the Comonotonic Independence, Kahneman and Tversky define the first of two theorems in their paper, the first of which states:

Theorem 1: (12)

$Suppose\ that\ sets\ of\ prospects\ (F^+, \succcurlyeq)\ and\ (F^-, \succcurlyeq)\ can\ each\ be\ represented\ by\ a\ cumulative\ functional.\ Then\ (F, \succcurlyeq)\ satisfies\ Cumulative\ Prospect\ Theory\ iff\ it\ satisfies\ double\ matching\ and\ Comonotonic\ Independence$

Where a cumulative function is defined as a function that transforms each probability separately such that the model transforms the entire Cumulative Distribution Function. This is done so that

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[27] $h(s) = f(s)$

[28] Tuthill, Jonathan, Frechette, Darren., "Non-expected Utility Theories: Weighted Expected, Rank Dependent, and, Cumulative Prospect Theory." St. Louis, Missouri., April 2002. P. 17. Accessed April 2016. http://ageconsearch.umn.edu/bitstream/19073/1/cp02tu01.pdf

events with extremely small probabilities are not overweighed like they were in Prospect Theory.[29] Cumulative Prospect Theory can only hold if double matching and Comonotonic Independence are satisfied.

Based on these various postulates, Cumulative Prospect Theory defines three versions of tradeoff consistency, which deal with preference relationships among different prospects in various states. First, they define Tradeoff Consistency (TC), which is satisfied by Expected Utility theory:

$If\ every\ x, x', y, y' \in X, and\ every\ f, f', g, g' \in F, and\ s, t \in S, then\ let\ x \in X, f \in F\ and$

$s \in S, then\ x\{r\}f\ is\ the\ outcome\ that\ yields\ x\ in\ state\ r\ coinciding$

$with\ \ f\ in\ all\ other\ states.$

$So\ a\ preference\ relation\ satisfies\ Tradeoff\ Consistency\ if$:

$$x\{s\}f \succcurlyeq y\{s\}g,\ x'\{s\}f \succcurlyeq y'\{s\}g\ and\ x\{t\}f' \succcurlyeq y\{t\}g'\ implies\ x'\{t\}f' \succcurlyeq y'\{t\}g' \qquad (13)$$

This means that state *s* will yield result *x* for prospect *f*. Tradeoff Consistency implies that for various preference relationships to be satisfied then yield *x* in state *s* for prospect *f* is at least as preferred to *y* in state *s* for prospect *g*. In conjunction, yield *x'* in state *s* for prospect *f* must be at least as preferred to yield *y'* in state *s* for prospect *g*. In addition to that, state *t* must yield *x* for prospect *f'* which must be at least as preferred to yield *y* in the same state for prospect *g'*. Therefore, yield *x'* in state *t* for prospect *f'* must be at least as preferred to *y'* in state *t* for prospect *g'*. This is rather abstract, but it is complete and states the satisfactory and "rational" preference relationships for trades. Tradeoff Consistency is the condition that implies that a von Neumann and Morgernstern utility function cannot have preference reversals. It says that if an agent prefers one prospect to another in one state, they will prefer that same prospect in a different state as well.

[29] Kahneman, Daniel, Tversky, Amos., "Advances in Prospect Theory: Cumulative Representation of Uncertainty." *Journal of Risk and Uncertainty*, 1992. Vol. 5, 299

Tradeoff Consistency, although important, is violated by the Allais paradox described earlier. So, when Expected Utility theory does not hold, then one can consider the next tradeoff consistency to explain agent behavior. This tradeoff consistency is titled Comonotonic Tradeoff Consistency:

$$\textit{Comonotonic Tradeoff Consistency } (CTC) \textit{ holds whenever } TC \textit{ holds and the prospects } x\{s\}f, y\{s\}g, x'\{s\}f, \textit{and } y'\{s\}g \textit{ are pairwise comonotonic, with the prospects } x\{t\}f', y\{t\}g', x'\{t\}f' \textit{ and } y'\{t\}g'. \quad (14)$$

Essentially Comonotonic Tradeoff Consistency says that if in state *s* the yields *x, y, x',* and *y'* for prospects *f* and *g* (respectively) are comparable and non-decreasing with the outcomes in state *s* and similar yields for prospects *f'* and *g'* then prospects are comonotonic and satisfy Comonotonic Tradeoff Consistency. Therefore, whenever prospects are pairwise comonotonic and the value functions have particular yields that satisfy comonotonicity between pairwise decisions, then Comonotonic Tradeoff Consistency holds. Comonotonic Tradeoff Consistency is a weaker consistency than Tradeoff Consistency, but it helps to explain more preference relations in ambiguous decisions due to the fact that it only allows comparisons of decisions that are comonotonic, as opposed to all decisions.

The third tradeoff consistency defined by Kahneman and Tversky is noted as Sign-Comonotonic Tradeoff Consistency (SCTC):

$$\textit{Satisfied if } CTC \textit{ holds whenever yields } x, x', y, y' \textit{ are either all nonnegative or all nonpositive} \quad (15)$$

Sign-Comonotonic Tradeoff Consistency is the weakest of the three consistencies, and it is satisfied when all possible outcomes are all below or above an agents' reference point.

Since the various tradeoff consistencies have been defined, it is possible to define the second major theorem of Cumulative Prospect Theory:

Theorem 2: (16, a-c)

$a. EUT\ holds\ iff \succcurlyeq satisfies\ TC$[30]

$b. Cumulative\ Utility\ Theory\ holds\ iff \succcurlyeq satisfies\ CTC$[31]

$c. Cumulative\ Prospect\ Theory\ holds\ iff \succcurlyeq satisfies\ double\ matching\ and\ SCTC$

The second major theorem of Cumulative Prospect Theory essentially says that the model's independence axioms weaken itself depending on the tradeoff consistency that holds.

3.2: Maxmin Expected Utility

Maxmin Expected Utility was formalized by Itzhak Gilboa and David Schmeidler in 1989,[32] and the theory considers a closed, convex set $\mathcal{C}$ of probability measures[33] on state space *S*, and von Neumann-Morgenstern Expected Utility function $U(\cdot)$ and an act is evaluated according to:

$$W(f(\cdot)) = \min_{\mu \in \mathcal{C}} \int U(f(\cdot))d\mu \qquad (17)$$

Where W is a utility function applied to some act f[34] and is equal to the expected utility of some roulette lottery, where the integral is over the state of the world for each state where there exists a roulette lottery.[35,36] An act in the paper by Gilboa and Schmeidler is defined as "functions from states of nature applied on some prior [sic] over a set of 'deterministic outcomes.'" Where

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[30] Wakker, Peter., "Prospective Reference Theory: Toward an Explanation of the Paradoxes," *Journal of Risk and Uncertainty*., 1989, The Netherlands: Kluwere Academic Publishers
[31] Wakker, Peter., "Continuous Subjective Expected Utility with Nonadditive Probabilities," *Journal of Mathematical Economic*, 1989, Vol.18., 1-27
[32] Gilboa, Itzhak, Schmeidler, David. "Maxmin Expected Utility with Non-Unique Prior." *Journal of Mathematical Economics*, 1989, Vol. 18, 141-153
[33] Also known as "Priors" which they will be called for the remainder of this paper
[34] An act is defined the same as a prospect is above
[35] In this system, a roulette lottery is defined as a game of chance with "physical probabilities" associated with the outcomes, where each outcome is associated with a prize. Additionally, a "physical probability" is defined as a probability whose outcomes are objective and associated with random physical systems, such as roulette wheels** (18)
[36] Anscombe, F.J., Aumann, R.J., "A Definition of Subjective Probability." *The Annals of Mathematical Statistics*. March 1963, Vol. 34(1), 199-205
**Physical Probabilities: Hajek, Alan, Zalta, Edward., "Interpretations of Probability," *The Stanford Encyclopedia of Philosophy*., Winter 2012 Ed., Accessed May 2016 http://plato.stanford.edu/archives/win2012/entries/probability-interpret/

a deterministic outcome is defined as outcomes that are predetermined where the bettor in this roulette lottery knows the possible losses and gains, known as outcomes. This particular utility function was axiomatized by Gilboa and Schmeidler, where they maintain the axioms of Completeness, Transitivity, Continuity, and Non-Degeneracy, which were presented by von Neumann and Morgenstern and show the relationship between utility and preferences. However, Gilboa and Schmeidler weaken the Independence axiom and replace it with two other axioms, which state:

Certainty-Independence: $\forall\, acts\ f, g \in \Lambda, and\ h \in L_c\ and\ \forall\ \alpha \in (0,1]: f \succ g \iff \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$[37] (19)

Uncertainty Aversion: $\forall\, acts\ f, g \in \Lambda\ and\ \alpha \in [0,1]: f \sim g\ implies\ that\ \alpha f + (1-\alpha)g \succcurlyeq f$ (20)

These two axioms weaken the independence axiom and allow the Maxmin model to be more applicable, especially in Ellsberg lottery-type situations. Standard independence allows *h* to be any act within *L*, whereas the Certainty-Independence restricts this to only constant acts, known as L_c. Certainty-Independence is much simpler than the von Neumann and Morgenstern independence axiom and does not exclude hedging, where hedging is defined as the act of taking the position in one market to offset another. Since hedging is not excluded, Gilboa and Schmeidler explain it through the Uncertainty Aversion axiom, because the axiom of Uncertainty Aversion provides a detailed explanation of a utility function when there exists ambivalent preferences, which is why it can account for uncertainty averse situations such as those presented in the Ellsberg paradox.[38] Finally, the general Maxmin Expected Utility function, as described by Gilboa and Schmeidler, is:

[37] Assume non-empty sets, and L_c as the constant functions in L_0, also let L be a convex subset of Y^s, which includes L_c.***

[38] For an example in which the act of hedging is an important feature of the Uncertainty Aversion axiom see: Machina, Mark, Siniscalchi, Marciano. "Ambiguity and Ambiguity Aversion." To appear in: *The Handbook of the Economics of Risk and Uncertainty*, Accepted: June 2013, 20

$$W(\mathcal{F}(\cdot)) = \alpha * \min_{\mu \in \mathcal{C}} \int U(\mathcal{F}(\cdot))d\mu + (1-\alpha) * \max_{\mu \in \mathcal{C}} \int U(\mathcal{F}(\cdot))d\mu \quad (21)$$

This value based Maxmin function now is able to account for a range of ambiguous decisions larger than that of the von Neumann and Morgernstern model. This model says that agents behave as if there exists a set of possible probabilities with a fixed utility function in order to compute their expect utility for any act; then the agents take the lowest possible probability and apply it across a set of outcomes. This is also known as a weighted average of the most pessimistic and the most optimistic expected utility values.

3.3: Rank-Dependent Utility

Rank-Dependent Utility was formalized by Quiggin (1982)[39] which ranked events based on probability weights, where probability weights can be interpreted as the importance that agents weight some outcome x_j.[40] Rank-dependence is also used in Yaari's Dual Theory (1987), which is described in section 4.1 of this paper. Rank-dependence is also an important feature of Kahneman and Tversky's 1992 Cumulative Prospect Theory model.[41] Rank-Dependent Utility was developed as a "better" version of Expected Utility theory, and the Rank-Dependent Utility model does not overweight extremely unlikely events so that there are not violations of first-order stochastic dominance like in Prospect Theory.

The general weight model for Rank-Dependent Utility is defined as:[42]

$$\sum_{j=1}^{n} \pi_j U(x_j) \quad (22)$$

*** Gilboa, Itzhak., Schmeidler, David. "Maxmin Expected Utility with Non-Unique Prior." *Journal of Mathematical Economics*, 1989, Vol. 18, 143-144

[39] Quiggin, John., "A theory of anticipated utility." *Journal of economics behavior and organization*, 1982, Vol. 3(4), 323-343

[40] Diecidue, Enrico, Wakker, Peter., "On the Intuition of Rank-Dependent Utility." *The Journal of Risk and Uncertainty*., 2001, Vol. 23 (3), 283.

[41] Tuthill, Jonathan, Frechette, Darren., "Non-expected Utility Theories: Weighted Expected, Rank Dependent, and, Cumulative Prospect Theory." St. Louis, Missouri., April 2002. P. 17. Accessed April 2016. http://ageconsearch.umn.edu/bitstream/19073/1/cp02tu01.pdf

[42] Diecidue, Enrico, Wakker, Peter., "On the Intuition of Rank-Dependent Utility." *The Journal of Risk and Uncertainty*., 2001, Vol. 23 (3), 281-298.

Where U is the utility function, the π_j's are decision weights, which are nonnegative and sum to one, and the x_j's are the different outcomes for various lotteries. The assumption of rank-dependence is incredibly important for utility theory because it relaxes the independence axiom and assumes that agents are either pessimistic or optimistic. Pessimism and optimism are important in this case because of the tier valued system used to define agent behavior. The model to be shown displays pessimistic behavior in agents through the distribution function which assigns weights based on certain outcomes. Additionally, agents are assumed to evaluate lotteries based on the worst possible outcome.[43] This also is a similar phenomenon seen in the Maxmin Expected Utility model. In Maxmin Expected Utility agents behave as if there is a set of possible probabilities with a fixed utility function in order to compute their expected utility for any act. However, they [the agents] apply the probability from the act that yields the lowest expected utility. This, in conjunction, is similar to Rank-Dependent Utility because the agents ordering for events are based on their pessimism and optimism of certain outcomes, and this pessimism or optimism infers a logical deduction of the probability of certain events occurring.[44] Quiggin uses simple lotteries and weighting functions to define relationships between risky and non-risky prospects. He defines these as "certainty equivalences" or CE:

$$\begin{gathered}If\ outcome\ c \in X\ is\ a\ non-risky\ prospect\ in\ the\ set\ of\ non-risky \\ prospects, X, and\ risky\ prospect\ y \in Y, where\ Y\ is\ the\ set\ of\ all\ risky\ prospects, \\ and\ c{\sim}y\ then\ it\ is\ to\ be\ called\ a\ certainty\ equivalent\ (CE)\ of\ y, or{:}\, c = CE(y) \qquad (23)\end{gathered}$$

Therefore, for an outcome c that is non-risky and outcome y that is risky, if c is indifferent to y then it is called the certainty equivalent of y, hence $CE(y)$. The next important feature of the Rank-Dependent Utility model uses an axiom of independence, which is another modified version from the one originally published by von Neumann and Morgenstern. It is defined as:

[43] Diecidue, Enrico, Wakker, Peter. "On the Intuition of Rank-Dependent Utility." *The Journal of Risk and Uncertainty*., 2001, Vol. 23 (3), 285
[44] Dillenberger, David, Postlewaite, Andrew, Rozen., Karen., "Optimism and Pessimism with Expected Utility." February 2014, 1-19

RDU axiom of independence: $For\ some\ outcome\ x \in X\ and\ some\ probability,\ p \in [0,1]$

$If\ y_1 = \{x, p\}\ and\ y_2 = \{x', p\}\ and\ for\ each\ i = 1,2, \dots n,\ \exists$:

$$c_i = CE\left\{(x_i, x_i'); \left(\frac{1}{2}, \frac{1}{2}\right)\right\}, i \in 1,2 \dots, n, \qquad and$$

$$x_1^* = CE(y_1), x_2^* = CE(y_2).\ Then:$$

$$\{c; p\} \sim \left\{(x_1^*, x_2^*), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\} \tag{24}$$

All of this means that if each non-risky element, *c*, is indifferent to a 50-50 bet consisting of corresponding risky elements *x* and *x'* then the set *{c;p}* is indifferent to to a 50-50 bet consisting of the certainty equivalents *{x;p}* and *{x';p}*.[45] This is its way of solving the risky versus non-risky problem that is presented in the Ellsberg Paradox. Since the new axiom of independence was established, Quiggin makes two assumptions, which led to the primary achievements of the Quiggin paper. The assumptions are made for the sake of a utility function that preserves preference ordering, which then also preserve original preference relationships that are described by von Neumann and Morgenstern. The assumptions are:

$$A1: If\ x_1, \dots, x_n \in X,\ x_n P_{x_{n-1}} P, \dots, P_{x_1}\ and\ \textstyle\sum_i p_i = 1, then\ \{x; p\} \in Y$$

$$A2: If\ y \in Y\ then\ \exists\ some\ x = CE(y) \in X \tag{25 a-b}$$

The first assumption says that there exists some set of outcomes, *X*, and associated probabilities, *P*, with lotteries occurring (and their probabilities sum to one) and the set of the lotteries and probabilities together is the payoff set, namely *Y*. The second assumption says that there exists some elements of *X* and *Y* that are certainty equivalents of each other. These assumptions together imply that there is some set of outcomes, *X*, with associated probabilities, *P*, that, when evaluated by the agent, all belong to the set of all possible outcomes of *X* and *P* called set *Y*. Since these outcomes and probability mixtures exist within *Y* it is possible to say that there exists

[45] Quiggin, John., "A theory of anticipated utility." *Journal of economics behavior and organization*, 1982, Vol. 3(4), 332

some certainty equivalents of outcomes x_i to outcomes y_i. These two assumptions led to a very important three-part theorem, which was the major contribution of the Quiggin paper, it reads:

$T1\!: Suppose\ X\ and\ Y\ satisfy\ A1\ and\ A2, and\ P\ satisfies\ Quiggin's\ 4\ axioms$[46]

$including\ RDU\ Independence:$

$Then\ \exists\ a\ [utility] function\ V\!: Y \to \mathcal{R}^n, st:$

$i\!:\ V(y) \geq V(y'), iff\ y \succcurlyeq y'$

$ii\!: V(x;p) = \sum_i h_i(p)U(x_i)\ \ for\ some\ functions\ U\ and\ h,$

$where\ h(1) = 1, and\ h\left(\frac{1}{2},\frac{1}{2}\right) = \left(\frac{1}{2},\frac{1}{2}\right).$

$If\ the\ functions\ V\ and\ V' satisfy\ i\ and\ ii, then\ \exists$

$constants\ a\ and\ b, with\ a > 0,\ \ st:$

$$iii\!: V'(y) = aV(y) + b \tag{26}$$

This theorem states that if the first two assumptions are satisfied, as well as the priors satisfied by the Rank-Dependent Utility independence axiom and non-degeneracy, then there is a utility function V that maps the outcome set Y onto the set of real numbers. By doing this the utility function is able to satisfy three conditions depending on the prior, act, and state. The first states that V is a utility function. That is, if the function is applied to y and y', and y yields the same or higher utility, then y is as least as preferred as y'. The second part states that if a utility function is applied to the set *(x;p)* where p is a prior, then the yielded utility will be the result of two separate utility functions applied on the prior and some x_i within that set. Similarly, if the utility function for the prior is equal to one then the prior will just be a weight of one and if it is a 50-50 bet then the prior will be a 50-50 distribution. Statement three says that if the utility functions, V

[46] For full detail of axioms 1-3 (axioms 4 is RDU Independence), see: Quiggin, John., "A theory of anticipated utility." *Journal of economics behavior and organization*, 1982, Vol. 3(4), 331-332

and *V'*, which satisfy parts one and two of the theorem, then they are affine transformations of one another.[47]

In order to properly define the full Rank-Dependent model, it is necessary to elaborate on the application of decision weights. A decision weight is essentially how much an agent will focus on a given probability, so therefore it is a weight applied on that probability. Additionally, decision weights tend to overweight small probabilities and underweight moderate and high probabilities.[48] However, because the transformations occur with cumulative rather than individual probabilities, the problem of over and under weighted decision weights, as described by Quiggin, do not occur in the Rank-Dependent model. Thus the maximum outcome of a particular lottery depends on its probability and not the decision weights. If a lottery depends only on probability *p*, then its position based on rank will always be one, because then the outcome is the maximal outcome of the lottery. Therefore, if we define the function $\omega(p)$ as a decision weight and ω is strictly increasing, such that preference relations satisfy stochastic dominance, then define the full Rank-Dependent Utility model (RDU) as described by Quiggin in 1982:

If x_j is an outcome where $x_1 > \cdots > x_n$ then

$$RDU(p_1 x_1; \ldots; p_n x_n) = \sum_{j=1}^{n} \pi_j U(x_j) \qquad \text{(27 a.)}$$

Where for each π_j, we can define the function:

$$\pi_j = \omega\big(p_1 + \cdots + p_j\big) - \omega\big(p_1 + \cdots + p_{j-1}\big) \qquad \text{(27 b.)}$$

And, $\pi_1 = \omega(p_1)$.

This function can also be represented by Yaari's dual function. Therefore, with regards to the paradoxes explained earlier, the Rank-Dependent Utility model does not fail the various

[47] Also known as: There exists constants, namely *a* and *b*, where *a* and *b* are greater than zero, so that *V'* applied on decision *y* is the same as constant *a* applied on utility function *V* which is applied on decision set *y* plus *b*.
[48] Kahneman, Daniel, Tversky, Amos. "Advances in Prospect Theory: Cumulative Representation of Uncertainty." *Journal of Risk and Uncertainty*, 1992. Vol. 5, 298

paradoxes because of its weakening of the independence axiom and the theorem listed earlier. The replacement of the Rank-Dependent axiom of independence allows for an agent to have a preference for hedging: the 50-50 bet of two ambiguous acts, as they are in the Ellsberg Paradox, yields a completely objective lottery of a 50-50 probability mixture. Then this occurs because the decisions in the Ellsberg Paradox are negatively correlated, as they are not comonotonic events, where a similar result occurs when Cumulative Prospect Theory is applied.[49]

[49] Machina, Mark, Siniscalchi, Marciano. "Ambiguity and Ambiguity Aversion." To appear in: *The Handbook of the Economics of Risk and Uncertainty*, Accepted: June 2013, 22

Section 4: Challenges to Non-Expected Utility Theory and Rationality Assumptions

4.1 Aspirations and Obstacles

Rubinstein (1998)[50] considers three aspirations for economists:

- Construction of new theories of choice because there have been clear deviations in experiments from the rational man theory.[51]
- The refinement of the notion of choice, for example, models that include intertemporal variables, in order to understand when agents make such decisions.
- The transformations of the notion of equilibrium. This is due to the fact that agents have to make inferences about the environment in which they operate. Therefore, strategic interactions and rational expectations are based on assumptions that are not entirely correct.[52]

Rubinstein continues to state that the challenges for scholars of Bounded-Rationality are to actually include procedural aspects of decision making in specific economic theories. With regards to the first of his three aspirations for economics to overcome, one can say that within the literature for Non-Expected Utility theory and Bounded-Rationality models, there have been plenty of new theories that take into account the experimental data. In particular, Machina (2013)[53] explains twelve new models for Ambiguity Aversion and explains their inner workings with regards to the Ellsberg Paradox. To state things simply, it would be difficult to create one model that accounts for all [utility maximization] decisions made by agents. However, these two fields [Non-Expected Utility and Bounded-Rationality] have done relevant work with regards to creating models, which satisfy and axiomatize decisions, especially in lottery-type situations.

[50] Rubinstein, Ariel. *Modeling Bounded Rationality* (MIT Press: 1998)
[51] Edwards, Ward. "The Theory of Decision Making." *Psychological Bulletin*, 1954, Vol 51(4), 380-417
[52] This relates to equilibrium because if decision makers are making decisions based on their preferences by analyzing the environment around them then those decisions directly effect the equilibrium in the economy.
[53] Machina, Mark, Siniscalchi, Marciano. "Ambiguity and Ambiguity Aversion." To appear in: *The Handbook of the Economics of Risk and Uncertainty*, Accepted: June 2013

The models created are not overarching models for the entire population, but they help to explain different types of preferences in different agents.

It is, however, important to show that paradoxical situations, such as the Ellsberg and Allais situations, can actualize in real world applications and not just in a laboratory environment. A paper from Suhonen, Sasstamoinen, and Linden (2014)[54] shows empirical observations from real life gambling markets, which correspond to the Allais experiment. They develop a model, which appends Yaari's (1987)[55] Dual Theory model with probability weights using Prelec's (1998)[56,57] function. In order to understand Yaari's dual theory, it is necessary to understand the Dual axiom of independence:

$$\forall a \in [0,1],\ G \succcurlyeq G', implies\ aG \boxplus (1-a)H \succcurlyeq aG' \boxplus\ (1-a)H \qquad (28)$$

Where *G*, *G'*, and *H* are Decumulative Distribution Functions (DDF's), *t* is a state, and *a* is a weight. Where DDF *G* is defined as:

$$G(v) = \Pr\{v > t\}\ and\ 0 \le t \le 1 \qquad (29)$$

Such that integrating $G_v(t)dt$ over the definite [0,1] integral gives the expected value of *v*. Doing this restricts the value of *v* and implies that no gambles can be made which might result in a loss greater than one's total wealth. Simultaneously, no gambles exist offering prizes larger than some predetermined number, therefore the gambles are all normalized. This is different from a Cumulative Distribution Function (CDF) in that a CDF will describe the likeliness of the probability of a random variable occurring that is smaller than some value, *t*. Whereas the DDF describes the probability of some random variable that will be greater than some value, *t*.

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54 Suhonen, Niko, Sasstamoinen, Jani, Linden, Mikael. "Are Gambling Behaviour and Allais Paradox Two Sides of the Same Coin? Evidence from Horse Race Betting." *University of Eastern Finland*, 1-26

55 Yaari, Menahem. "The Dual Theory of Choice under Risk." *Econometrica*, Jan. 1987, Vol. 55(1), 95-115.

56 Prelec, Drazen., "The Probability Weighting Function." *Econometrica*, May 1988, Vol. 66(3), 497-527

57 Neilson, William, Stowe, Jill., "A Further Examination of Cumulative Prospect Theory Parameterizations." *The Journal of Risk and Uncertainty*, 2002, Vol. 24(1), 31-46

This is important because it states that if functions are continuous then *G(t-)=G(t)*. This also implies that DDF $\hat{G}(t) = G(t)$. If they are not continuous functions, then the space in-between the two DDF's is the *x* interval that is defined in earlier axioms of Yaari's paper. In this independence axiom, the mixture operator, ⊞, is defined as:

$$aG \boxplus (1-a)H = (aG^{-1} + (1-a)H^{-1})^{-1} \tag{30}$$

Meaning that, the mixture operator takes the inverse of DDF's *G* and *H*, then takes the mixture of the two DDF's, and then takes the inverse of the DDF's *G* and *H* again after the mixture operation has occurred. This makes it so that the mixture of the DDF's is taken horizontally and not vertically, which is important if the DDF's are not continuous.

This independence axiom is very similar to that of Comonotonic Independence from Kahneman and Tversky, in that it uses comonotonic prospects to define the independence of decision making in lottery-type situations. Yaari's theorem then states, based on the von Neumann and Morgenstern axioms, with the dual independence axiom in place of the original independence axiom, that preference relationships satisfy these axioms if and only if there exists a continuous and nondecreasing real function, *f*, defined on the unit interval, *u*, for all *u* and *v*, where *v* is a random variable representing a simple lottery, such that:

$$u \succcurlyeq v \Leftrightarrow \int_0^1 f\big(G_u(t)\big)dt \geq \int_0^1 f\big(G_v(t)\big), where\ t\ is\ a\ outcome \tag{31}$$

This function is the representative theorem of Yaari's 1987 paper. It states that a random variable, *u*, which is between zero and one, is at least as preferred as random variable *v*, if and only if, from the definite integral between zero and one, the resulting function multiplied by the DDF G_u for outcome *t* is greater than the same result between zero and one for DDF, G_v. As the paradoxes stands, the dual independence axiom says that no gambles that can be made which result in a loss greater than total wealth, and that no gambles exist which offer prizes larger than some predetermined number, which is known to the agent, and then all gambles are normalized through the DDF.

In order to fully understand the obstacles implied through the Suhonen, Sasstamoinen, and Linden paper, it is necessary to define Prelec's function, which states:

$$g(p) = e^{-(-lnp)^{\gamma}} \quad [58] \tag{32}$$

Where the *g* is one decision from the set *G* in Yaari's Dual Theory model. What is interesting about Prelec's function is that a decrease in gamma allows it to become more convex to the right of $1/e$, which furthers the point by Prelec that as the amount of returns on lottery situations increases, the probability weighting function changes. This leads to the generalized form for the model used by Suhonen, Sasstamoinen, and Linden which is represented by:

$$V(P) = u(r_0) + \sum_{j=1}^{n} w(p_j)[u(r_j) - u(r_{j-1})] \tag{33}$$

Where *u* is the utility function, *w* is the weighting function, which is strictly increasing from [0.1] to [1]. The model is Rank-Dependent,[59] becoming the Expected Utility model when the weight function is linear, and the Dual Theory model when the utility function is non-linear. They then proposed an alternate model for these real world applicable lottery situations. The model is a variant of the general Rank-Dependent Utility model. Suppose the Uncertainty Function is modeled with an increasing *U*, and continuous mapping U: $[0,1] \longrightarrow [0,1]$ with *U(0) = 0* and *U(1)* = 1, and *r* being the list of all *r's* within outcome set *R,* where outcomes are listed from worst to best; for example, r_1 is the worst outcome and r_n is the best outcome. Therefore, the model is:

$$V(P) = r_0 + \sum_{j=1}^{n} U(p_j)[r_j - r_{j-1}] \tag{34}$$

This particular Uncertainty Function is important for modeling real world uncertainty situations because it combines aspects of risk behavior and probability weighting. Their model uses uncertainty through modeling simple lotteries that have uncertain outcomes (in this case, horse race betting behavior.) Their results state that bettors' behavior is invariant in time and in the scale of betting. All that matters are the attitudes towards risk and uncertainty. Their model

[58] See footnote 56
[59] Where Rank-Dependence is defined as it is in section 3.3 of this paper

also shows that in real life decisions they were able to find the same preference patterns observed by Allais in 1953. More amazingly, they were able to find these results in a real-world situation and not in a laboratory, which was one of the criticisms of the Allais experiment from Grether and Plott in 1979.[60] Not only were these Allais-type results, but based on the data and analysis, the results were also statistically significant. Their results show that gamblers are not only risk-averse, but also prone to misperceptions of probabilities: over-weighing low probabilities and under-weighing high probabilities. This shows that people are very averse to uncertainty, as opposed to just slightly averse to it, like it is presented in the Dual Theory model.

Their paper on horse race betting and the Allais paradox was published in 2012, which was almost 60 years after the Allais paper. The Allais paradox has received harsh criticism in the past; however, no one has been able to offer an actual approach that solves the paradox in terms of von Neumann-Morgenstern Expected Utility theory. Therefore, the question becomes, how do Non-Expected Utility theories and theories of Bounded-Rationality expect to find solutions to various paradoxes that have challenged rationality and its framework, on which economists have built their models? One way is to incorporate the ideas of Rabin, who suggests that there is potential for using Neoclassical optimization models and psychological models to bound rationality within utility models. Although suggesting the input of psychological modeling into economics is considered an unforgivable sin to most Neoclassical economists, Rabin states,

"Other examples of human limits to rationality, I will argue, are genuinely best understood in terms of optimization models. As surely as optimization captures cases of fully rational behavior, it captures the psychology and resulting behavior of some limits to rationality."[62]

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[60] Grether, David, Plott, Charles. "Economic Theory of Choice and the Preference Reversals Phenomenon." *The American Economic Review*, Sept. 1979, Vol. 69(4), 623-638.

This statement is incredibly important because it means, in essence, that Bounded-Rationality models can use optimization models to study human behavior, but those optimization models also include psychological data, which economists generally do not interpret. Rabin continues, "Many of the ways humans are less than fully rational are not because the right answers are so complex. They are instead because the wrong answers are so enticing."[61] In this statement, I would say that agents are not fully rational not just because the wrong answers are so enticing, but at given moments, in given situations, and because of limits to information, agents are not fully rational. It is not possible to blame agents for not being "fully rational," due to the limits to information and time constraints, such that being "fully" rational (whatever that may entail) is nearly impossible. This does not necessarily mean that agents are irrational, it simply means that agents are not perfect and at some point the amount of time it takes to gather all the relevant information before making a decision can be costlier than the actual decision. Since systematic errors occur frequently in optimization models, same as they do for agents in their daily life, it would be optimal for economists to capture these systematic errors and incorporate them into models.[62]

4.2 Rationality and Neoclassicism

Neoclassical economists have repeatedly attempted to describe possible issues with the ways in which Allais and Ellsberg went about their experiments which lead to the founding of these paradoxes. The most famous example being Grether and Plott in 1979, who, despite displaying major arguments against the paradoxes, recreated the experiment taking into account their own theories of why Allais and Ellsberg could be wrong, and still found preference reversal behavior. Grether and Plott outline two questions that are of particular interest and need to be answered in order for these psychological experiments to be seen as relevant to many

[61] Rabin, Matthew. "Incorporating Limited Rationality into Economics." *Journal of Economic Literature 2013*. Vol. 51(2): 529

[62] This is different from uncertainty because in uncertain lottery-type situations the agent has all the information about the associated probabilities and outcomes, whereas here, for some decisions, agents do not have all of the information necessary to make a "rational" decision.

economists. The first, "Does the phenomenon exist in situations where economic theory is generally applied?"[63] The answer to this question is apparently "yes," given that other paradoxes to Expected Utility theory have developed since the formulation of the Allais Paradox in 1953. An example is the paper explained above that shows Allais-type preference reversal behavior in horse race betting. Therefore, there has been significant work done to show that the phenomenon does exist in places that utility theory generally applies. The second question that Grether and Plott ask, "Can the phenomenon be explained by applying standard economic theory or some immediate variant thereof?" This question is much more complex to answer. It is difficult to answer this question as "yes" or "no;" since, the literature has proven more sophisticated. Some economists, such as Grether and Plott, would claim that Non-Expected Utility theory is an extension of Neoclassical Utility theory. Therefore, these types of economists would say that this phenomenon could be explained through the application of standard economic theory. Some other economists, possibly Rabin,[64] may claim that the answer would be the same as Grether and Plott's answer, but for Bounded-Rationality models. Even though Bounded-Rationality models are specified in the behavioral field, some may say that they are extensions of von Neumann-Morgenstern Expected Utility theory, and are therefore neoclassical. However, many, such as Winter,[65] may answer "no" to the above question, and, economists who subscribe to a similar school of economic thought as Winter may make the claim that Neoclassical utility theory has failed to provide models that explain preference reversal behavior. Despite the great differences between these economists, it is important to note that each branch of economic thought has made important contributions to utility theory.

[63] Grether, David, Plott, Charles. "Economic Theory of Choice and the Preference Reversals Phenomenon." *The American Economic Review*, Sept. 1979, Vol. 69(4), 624.
[64] See footnote 4
[65] See footnote 6

This debate is likely to continue, but it is key to note what can be defined as rational. Edwards (1954) states that the "Economic man has three particular qualities."[66] The first quality says that homo economicus (the economic man) has all information, which is obviously a super-human and impossible effort, which is what Edwards was attempting to convey. The second quality is that homo economicus is infinitely sensitivity, meaning that it can be assumed that the alternatives available to any individual are continuous. Simultaneously, there exists infinitely divisible functions, prices are infinitely divisible, and there are no transaction costs to agents' decisions, in that homo economicus is sensitive to all of these functions, decisions, and divisibility of these outcomes. Finally, Edwards defines rationality as meaning two things: First, that homo economicus can weakly order the states that he receives, and second, he can (and does) make choices to maximize something.

Edwards sees the major problem with the rationality argument is the axiom of transitivity of preferences. Consequently, transitivity and independence are the two major axioms that have been shown not to hold in several situations. Hodgson (2012)[67] claims that the fault in utility maximization models, whether expected or non-expected, is the universal applicability for which economists strive. The quest for this generality greatly limits models, according to Hodgson (and Rodrik), and it is because of the counter examples that game theory models have contributed that traditional rationality and Neoclassical Expected Utility models have been greatly challenged. There are several instances where game theory has challenged aspects of rationality in economics. For example, viewing the Ellsberg paradox in a game theory format has challenged various Non-Expected Utility theories.[68] Machina and Siniscalchi (2013) view this phenomenon and show that the Choquet model is "incapable of exhibiting such a

[66] Edwards, Ward. "The Theory of Decision Making." *Psychological Bulletin*, 1954, Vol 51(4), 380-417

[67] Hodgson, Geoffrey. "On the Limits of Rational Choice Theory." *Economic Thought*, 2012, Vol 1, 94-108

[68] Machina, Mark, Siniscalchi, Marciano, (2013), et al.

preference" despite the model evaluating the bets according to the Choquet model.[69] Hodgson states that game theory models have challenged the 'as if' of rationality. This implies that agents act 'as if' they and other agents in a market are rational. Hodgson's major problem with this 'as if' assumption is that it requires economists to "treat individuals as capable of emulating incredible super-calculators with unbounded cognitive capacities, without any consideration of how they would manage to do this."[70]

[69] Machina, Mark, Siniscalchi, Marciano. "Ambiguity and Ambiguity Aversion." To appear in: *The Handbook of the Economics of Risk and Uncertainty*, Accepted: June 2013, 48
[70] Hodgson, Geoffrey. "On the Limits of Rational Choice Theory." *Economic Thought*, 2012, Vol. 1, 100

Section 5: A New Perspective on the Development of Utility Theories

5.1 The Problem with Linearity of the Development of Utility Models

Rodrik's, *Economics Rules* details many of the problems with the thinking of economists. Rodrik does this by presenting many examples in Macroeconomics where the field has either gained immense generality or there is an inherent problem with the fact that economists apply models within in the first steps of examining a problem, as opposed to doing a more complete analysis before applying a model to a situation. Rodrik also says that the major reason economists separate themselves from other social sciences is because economists have models and use mathematics; however, models are the framework to yield solutions for many situations. For utility theorists, such as those described above, utility theory has been seen as a field that has moved linearly, and that is not the fault of utility theory, but a reflection of the general trend in economics. To be linear in development I mean that a model successfully describes agent behavior for some time and then a paradox is found and then a new model has to be found in order to account for the inconsistency of a present part of said model. Though this is true for most sciences, it has particularly plagued utility theory. That is exactly how the field has progressed linearly. This linear progression has has limited the scope of utility theory. The argument is that utility models need to be looked at as useful for specific situations. Analyzing a situation with a model already in mind can be helpful such that a situation can then be easily analyzed. Similarly, if a model can be applied quickly to a situation, then it is likely that the situation can be solved rather simply. However, it is not always the case that the model that is in mind is the correct model for a particular situation; thus, a model that has worked in the past for a different scenario, but in a different scenario an economist may forego some of the assumptions of the situation so that said model will work. This does not mean that economists need a new model for every situation; in fact, that would make the field overly complex, and the problem with the development of models would continue. This also does not mean that it is necessary for economists to create a generally encompassing model for the world, because doing that would be next to impossible, and a model of that size would be useless. However, in utility theory it seems that this is the direction that utility theorists are headed. This would be, as they say in the case of the cartographers in Borges, similar to a map that encompasses and is the size of an entire region, useless.

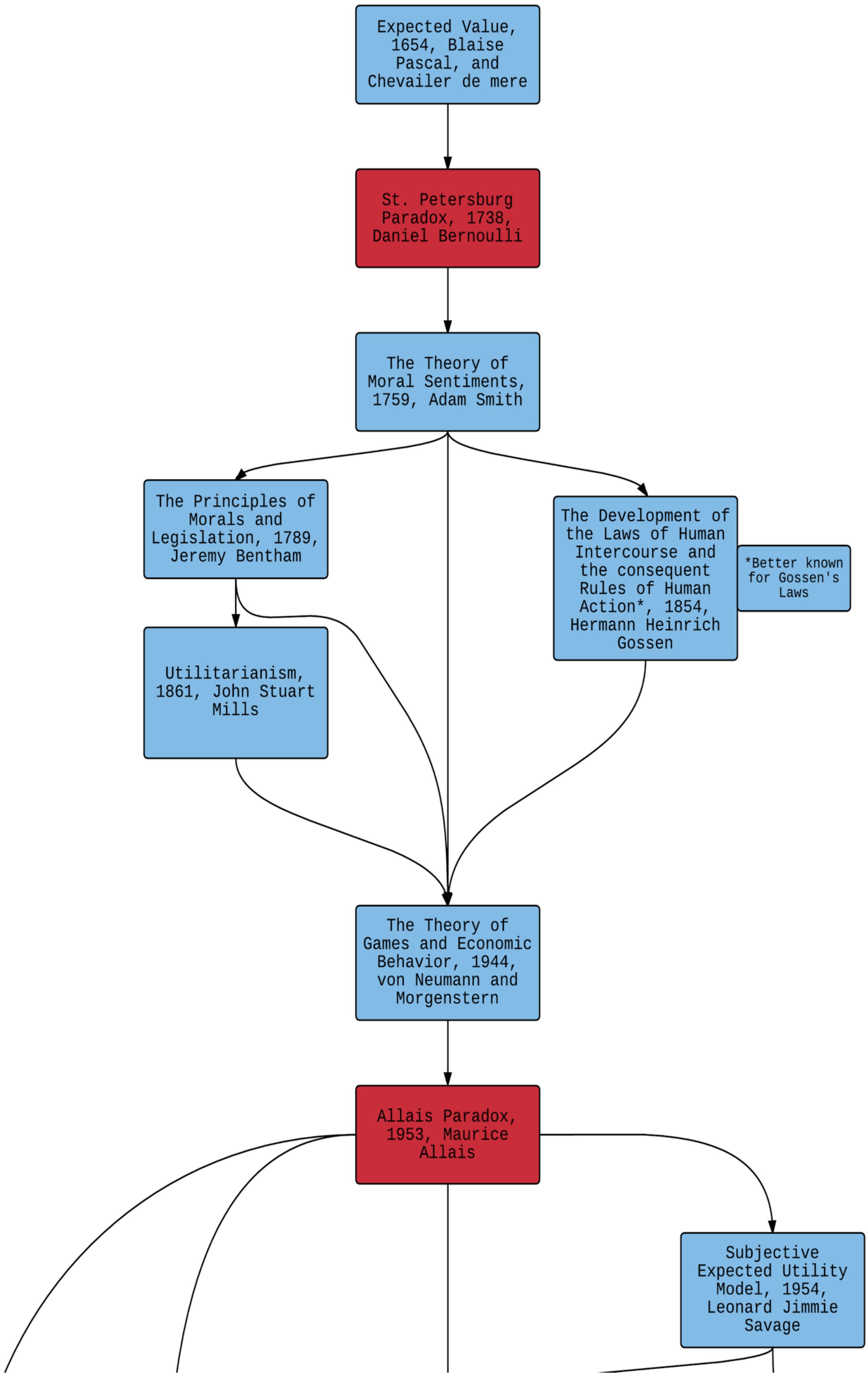
Rodrik discusses these points for Macroeconomics, but they can also be applied to utility theory. Linear developments in utility theory have historically helped the beginnings of the evolution of economic thought. *The Theory of Moral Sentiments* by Adam Smith in 1759 related the notion of utility to the overall behavior of the rich and poor in society. This relation was important for the development of economic thought because it influenced a host of different thinkers in economics ranging from the marginalists to utilitarians. Through the development of these different types of schools of economic thought the idea of modeling behavior of agents came about. These developments have drastically changed given that now economists have complex mathematical models of utility theory. The changes that have evolved in utility theory and the progress that has been made is incredibly significant. However, for true progress to be made, economists need to recognize the different ways that utility models have developed. Through changing the structure of the view of utility theory it will be easier to see the links between different models and therefore be easier to see the commonalities of the models and how utility theorists can exploit those similarities to create better models and better agent behavior. Similarly, this will also allow for more work to be done across different schools of economic thought in utility theory. This view will simultaneously help economists so that they do not attempt to create models that are so general that the model does not actually describe anything helpful.

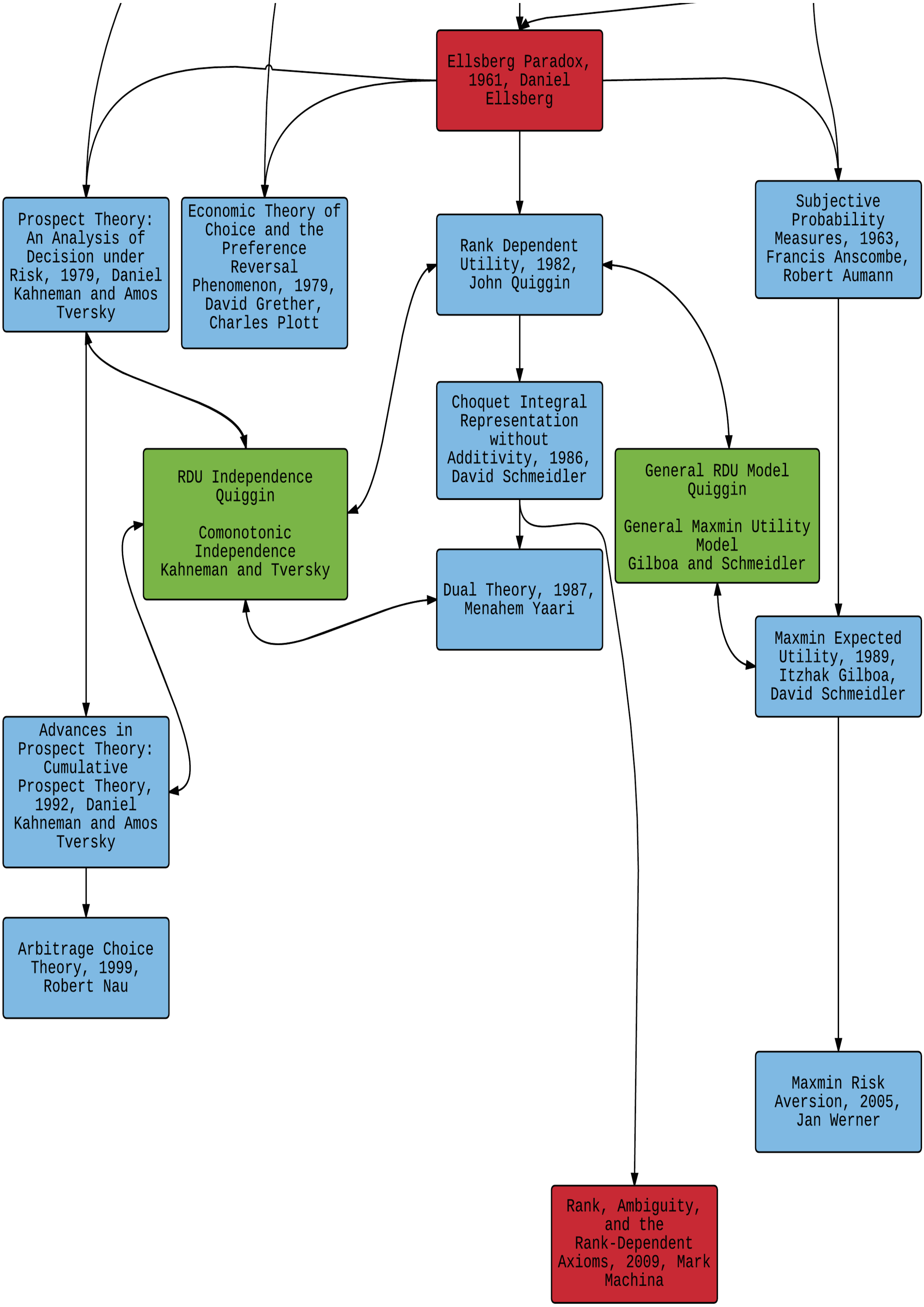
5.2 Development Tree of Utility Theories

The main contribution of this paper is to say that all of these models have their significant contributions, and they all have their uses, however, to classify any one of them as “the economic utility model” would be incorrect. Economists have strived to create the best models possible and in many ways they have succeeded, but it is important is to look at how these models work in various situations, and what the mathematics of them can tell economists about agent behavior in risk averse and ambiguous decisions. All these models tell economists something different, and their uses are all pertinent. Utility theorists need to stop viewing utility theory as a paradoxical world that needs to be solved, but instead as a world that is incredibly complex and that there are different solutions for different worldly occurrences. This goes for different behaviors for utility maximization, different classes, countries, and agents.

Additionally, as useful as the theories might be, it is important to recognize that every agent will not fit into a mathematical equation. Due to the current view of development, utility theory has progressed linearly, as a progression of model and paradox, but instead I provide a chart proposing how economists should view the developments of utility theory. This chart is different because it recognizes how the different branches of utility theory have changed and evolved because of each other. The present evolutional view is the linear development of utility theories. However, the chart that I provide describes various utility models in economics and shows how they are built from each other, despite belonging to different fields of economic thought. By doing this, I am able to show the dominant model(s) in each field and represent a paradox that may have caused a failure for a particular utility model in a particular situation. Though this tree is obviously not complete, it does show the basic models and some lesser known models for various branches of utility theory. The diagram is split into three different branches of utility theory: The Empirical, the Abstract, and the Subjective, respectively from left to right. Further down the chart we see that the independence axiom similarities between Quiggin's Rank-Dependent utility model and Cumulative Prospect Theory links these two branches of economic thought. Similarly, I show that agents acting pessimistically relate the Maxmin model to the Rank-Dependent model. Through defining pessimism and optimism as a property that is not typically found in Neoclassical Utility models, I claim that Subjective developments of utility models are, albeit closely relate, different from neoclassical development. This shows that all three branches are closely linked to each other through their axioms and equations. The paradoxes are in red, and the links between the branches are in green.[71]

[71] Created by author


Expected Value, 1654, Blaise Pascal, and Chevailer de mere
St. Petersburg Paradox, 1738, Daniel Bernoulli
The Theory of Moral Sentiments, 1759, Adam Smith
The Principles of Morals and Legislation, 1789, Jeremy Bentham
The Development of the Laws of Human Intercourse and the consequent Rules of Human Action*, 1854, Hermann Heinrich Gossen
*Better known for Gossen's Laws
Utilitarianism, 1861, John Stuart Mills
The Theory of Games and Economic Behavior, 1944, von Neumann and Morgenstern
Allais Paradox, 1953, Maurice Allais
Subjective Expected Utility Model, 1954, Leonard Jimmie Savage



Ellsberg Paradox, 1961, Daniel Ellsberg
Prospect Theory: An Analysis of Decision under Risk, 1979, Daniel Kahneman and Amos Tversky
Economic Theory of Choice and the Preference Reversal Phenomenon, 1979, David Grether, Charles Plott
Rank Dependent Utility, 1982, John Quiggin
Subjective Probability Measures, 1963, Francis Anscombe, Robert Aumann
Choquet Integral Representation without Additivity, 1986, David Schmeidler
RDU Independence Quiggin
Comonotonic Independence Kahneman and Tversky
General RDU Model Quiggin
General Maxmin Utility Model Gilboa and Schmeidler
Dual Theory, 1987, Menahem Yaari
Maxmin Expected Utility, 1989, Itzhak Gilboa, David Schmeidler
Advances in Prospect Theory: Cumulative Prospect Theory, 1992, Daniel Kahneman and Amos Tversky
Arbitrage Choice Theory, 1999, Robert Nau
Maxmin Risk Aversion, 2005, Jan Werner
Rank, Ambiguity, and the Rank-Dependent Axioms, 2009, Mark Machina

5.3 A New View of Development

If economists were to view all utility theories strictly as direct extensions of Neoclassical Utility theory, then they would miss many important developments of utility theory. Viewing these paradoxes and the models in a "tree" type development, makes it easy to see that there are successes and failures in each branch and it is important to use the successes of the various branches. By viewing the tree as three different branches, the Empirical, the Abstract, and the Subjective[72] economists are able to take the successes from the various branches and have a wide variety of models for the apparent wide variety of lottery-type situations. It is shown that the connections between the branches allows for progress in utility theory, and that is why the major contribution of this paper is to show the connections between these branches. As stated and shown, without the development of Quiggin's Rank-Dependent Model the furthered development of Cumulative Prospect Theory would have been much more difficult. However, it is because of Anscombe and Aumann that Maxmin Expected Utility was created by Gilboa and Schmeidler. Additionally, because of rank-dependent aspects and the contribution from Quiggin with regards to pessimism and optimism describing how agents rank probabilities assigned to outcomes, it is shown that pessimism and optimism leads to the assumption from Quiggin which assumes that agents apply the lowest ranking probability to all possible outcomes. These connections between the fields of utility theory are what make progress happen. By acknowledging the connections and showing that all these models are related, it can be said that the most significant progress can only be made through significant research outside of a particular subfield of utility theory.

Rodrik lists several commandments for economists at the end of his book *Economics Rules*, I have taken five of his commandments and direct these ones specifically for utility theorists:

1. Economics is a collection of models; cherish their diversity.

[72] As in subjective probabilities

2. It's a model, not *the* model.
3. Make your model simple enough to isolate specific causes and how they work, but not so simple that it leaves out key interactions among causes.
4. Unrealistic assumptions are OK; unrealistic *critical* assumptions are not OK.
5. To map a model to the real world you need explicit empirical diagnostics, which is more craft than science.

Section 6: Conclusion

Economics is a massive field. It is a social science that has seen explosive development within the last 100 years, and it is a field that influences all other social sciences. With this in mind, it is important to remember that before Adam Smith's famous *The Wealth of Nations* came *The Theory of Moral Sentiments*. The beginnings of utility theory are important and so are the developments from the 19th century with the emergence of marginalists and utilitarians, which led to 20th and 21st century utility theory. The thinkers of the 20th century created a broad field that has defined many important models for utility theorists, and this micro thinking has even greatly influenced macroeconomists.

In short, utility models are incredibly important and their developments are also essential to understand the history of modern economic thought and the future development of utility modeling. For this particular reason, I have provided the above flow chart of utility models, their developments, connections, as well as the list of commandments from Rodrik. If economists stop viewing the development of utility theory as linear, then they can start to take the benefits from each branch of utility theory and apply that to the failures of other utility models. This approach yields a more concrete and complete view of utility modeling and will be more beneficial for the future development of utility models. Competition of different branches of utility theory and other sciences leads to progress; it is important to recognize the work of other branches and find places to improve. What the behavioralists may call irrational an optimization economist may call constrained optimization, and the meeting point between them is Bounded-Rationality. When I discuss the development of utility theory as linear, what I mean is the "paradox, model…" type development and how harmful this view can be. So much so that Rodrik wrote a book about it, I am simply offering a warning to utility theorists, providing a solution, and some of the connections between the models, both mathematically and intuitively. The three types of schools of utility theorists which I describe each have their various uses in particular situations. For example, behavior models are very useful where there is data present, such as data derived derived from behavioral experiments similar to the data generated in the Ellsberg and Allais paradoxes, and when economists attempt to justify utility models, which are typically used to understand conceptual agent behavior.[73] In conjunction, models that use subjective probabilities are very useful for understanding conceptual agent behavior through the use of probability weights[74] and agents beliefs of pessimistic versus optimistic outcomes. Similarly, Neoclassical utility theory has its place in defining agent rationality and making axioms to conceptualize the reasoning behind agents preferences.[75]

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[73] Darnton, Andrew, "An overview of behavior change models and their uses." *Centre for Sustainable Development, University of Westminster*, July 2008, 34
[74] Hampton, J. M., Moore, P.G., "Subjective Probability and its Measurement." *Journal of the Royal Statistical Society*, Vol. 136(1), 1973., 21-42
[75] Simon, Herbert., "Rationality in Psychology and Economics." *The Journal of Business*, Vol. 59(4), Oct. 1986., S210

Appendix of Definitions, Formulas and Models:

(1) Expected Utility, Daniel Bernoulli:

$\sum_{i=1}^{n}(p_i)U(\bar{x}_i)$

(2) von Neumann and Morgenstern axioms and general terminology for Preference Relations:

A preference relation is denoted as $\succcurlyeq$, (which is a binary relation on the set of alternatives, $X \subset \mathcal{R}^n$, allows us to compare outcomes) where $x, y \in X \; and \; x \succcurlyeq y$ is read as "x is at least as good as y." Similarly, $x \succ y$ is read as "x is strictly preferred to y." And, $x{\sim}y$ is read as "x is indifferent to y."

(a) Completeness: $\forall x, y \in X, we \; have \; x \succcurlyeq y \; or \; y \succcurlyeq x \; or \; both \; (both \; implies \; x \sim y)$

(b) Transitivity: $\forall x, y, z \in X \; if \; x \succcurlyeq y \; and \; y \succcurlyeq z, then \; x \succcurlyeq z$

(c) Independence: $The \; preference \; relation \; \succcurlyeq$

$on \; the \; space \; of \; simple \; lotteries, \Lambda, satisfies \; the \; independence \; axiom \; if \; \forall \; \lambda, \lambda', \lambda'' \in$ Λ and $\alpha \in (0,1)$, we have:

$\lambda \succcurlyeq \lambda' iff \; \alpha\lambda + (1-\alpha)\lambda'' \succcurlyeq \alpha\lambda' + (1-\alpha)\lambda''$

(d) Continuous: $The \; preference \; relation \succcurlyeq$

$on \; the \; space \; of \; simple \; lotteries \;\; is \; continuous \; if \; for \; any \; \lambda, \lambda', \lambda'' \in \Lambda, the \; sets$:

$\{\alpha \in [1,0]: \alpha\lambda + (1-\alpha)\lambda' \succcurlyeq \lambda''\} \subset [0,1]$

and

$\{\alpha \in [1,0]: \lambda'' \succcurlyeq \alpha\lambda + (1-\alpha)\lambda'\} \subset [0,1]$

$are \; closed$

(3) Simple lotteries: $\lambda \; is \; a \; list \; of \; probabilities \; associated \; with \; a \; set \; of \; outcomes, X,$

$\lambda = (p_1, \ldots, p_N) \; with \; p_N \geq 0 \; \forall \; n, and \; \Sigma_n p_n = 1, \qquad And \; there \; is \; some \; set \; of \; outcomes,$

$X = (x_1, \ldots, x_n), each \; of \; which \; occurs \; with \; some \; known \; probability, p_i \in \lambda$

(4) Closed sets:

$Fix\ a\ set\ \Omega \subset \mathcal{R}^n.\ A\ set\ B \subset \Omega\ is\ closed\ iff\ for\ every\ sequence$

$\omega^m \to \omega \in \Omega, with\ \omega^m \in B\ \forall\ m, we\ have\ \omega \in B$

(5) General form for the Prospect Theory model:

$\Sigma_{i=1}^n v(x_i) * \pi(p_i)$

(6) First-order Stochastic Dominance

$Let\ \ \tilde{x}_a\ and\ \tilde{x}_b\ be\ random\ variables\ with\ cumulative\ distribution\ functions$

$F_a(x)\ and{:}\ F_b(x).\ \ Then\ F_a\ first-order\ stochastically\ dominated\ F_b\ if\ F_a(x) \leq F_b(x)$

(7) Strict Monotonicity:

$Strict\ Monotonicity{:}\ Preference\ Relation \succcurlyeq on\ X\ is\ monotonic\ if\ \forall\ x, y \in X\ and\ y \gg$

$x\ implies\ y \succ x.\ It\ is\ strictly\ monotonic\ if\ y \geq x\ and\ y \neq x\ imply\ that\ y \succ x$

(8) Comonotonicity:

$Comontonicity{:} Prospects\ f\ and\ g\ are\ comonotonic\ \ if\ there\ are\ no\ pairs\ of\ states,$

$s, s'\ such\ that{:}\ f(s) \succ f(s')\ and\ g(s) \preccurlyeq g(s')$

(9) Comonotonic Independence:

$Comontonic\ Independence\ requires\ that\ f \succ g$

$\Longrightarrow \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$

$\forall \alpha \in (0,1)\ and\ all\ f, g, h\ that\ are\ pairwise\ comonotonic$

$Where\ Pairwise\ Comonotonic\ is\ defined\ as\ comparisons\ between\ preference$

$relations\ that\ are\ comonotonic$

(10) Double Matching:

$\forall\, Prospects\ f, g \in F, if\ f^{+} \approx g^{+}\ and\ f^{-} \approx g^{-}, then\ f \approx g,$

$where\ f^{+} and\ g^{+} \in \mathcal{R}^{+}\ and\ f^{-}\ and\ g^{-} \in \mathcal{R}^{-}$

(11) Prospects (Acts):

$F = \{f : S \to X\}, where\ F\ is\ the\ set\ of\ all\ uncertain\ prospects, f.\ \ S\ is\ a\ finite\ set\ of$

$states\ of\ nature, and\ X\ is\ the\ set\ of\ all\ possible\ outcomes.$

(12) Theorem 1: Kahneman and Tversky:

$Suppose\ (F^{+}, \succcurlyeq)\ and\ (F^{-}, \succcurlyeq)\ can\ represent\ a\ cumulative\ function.$

$Then\ (F, \succcurlyeq)\ satisfies\ Cumulative\ Prospect\ Theory\ iff\ it\ satisfies\ double\ matching$

$and\ Comonotonic\ Independence$

(13) Tradeoff Consistency:

$If\ every\ x, x', y, y' \in X, and\ every\ f, f', g, g' \in F, and\ s, t \in S, then\ let\ x \in X, f \in F\ and$

$s \in S, then\ x\{r\}f\ is\ the\ outcome\ that\ yields\ x\ in\ state\ r\ coinciding$

$with\ \ f\ in\ all\ other\ states.$

$So\ a\ preference\ relation\ satisfies\ Tradeoff\ Consistency\ if$:

$x\{s\}f \succcurlyeq y\{s\}g,\ x'\{s\}f \succcurlyeq y'\{s\}g\ and\ x\{t\}f' \succcurlyeq y\{t\}g'\ implies\ x'\{t\}f' \succcurlyeq y'\{t\}g'$

(14) Comonotonic Tradeoff Consistency:

$Comonotonic\ Tradeoff\ Consistency\ (CTC)\ holds\ whenever\ TC\ holds\ and\ the\ prospects$

$x\{s\}f, y\{s\}g, x'\{s\}f, and\ y'\{s\}g\ are\ pairwise\ comonotonic, as\ well\ as\ the\ prospects$

$x\{t\}f', y\{t\}g', x'\{t\}f' and\ y'\{t\}g'.$

(15) Sign-Comonotonic Tradeoff Consistency:

$Satisfied\ if\ CTC\ holds\ whenever\ yields\ x, x', y, y'\ are\ either$

$all\ nonnegative\ or\ all\ nonpositive$

(16) Theorem 2: Kahneman and Tversky:

$a. EUT\ holds\ iff\ \succcurlyeq\ satisfies\ TC$

$b. Cumulative\ Utility\ Theory\ holds\ iff\ \succcurlyeq\ satisfies\ CTC$

$c. Cumulative\ Prospect\ Theory\ holds\ iff\ \succcurlyeq\ satisfies\ double\ matching\ and\ SCTC$

(17) Maxmin Expected Utility Function:

$$W\big(f(\cdot)\big) = \min_{\mu \in C} \int U\big(f(\cdot)\big) d\mu$$

(18) Roulette Lottery:

In this system, a roulette lottery is defined as a game of change with "physical probabilities" associated with the outcomes, where each outcome is associated with a prize. Additionally, a "physical probability" is defined as a probability whose outcomes are objective and associated with random physical systems, such as roulette wheels

(19) Certainty-Independence:

$\forall\ acts\ f, g \in \Lambda, and\ h \in L_c\ and\ \forall \alpha \in [0,1]: f \succ g\ \Leftrightarrow \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$

(20) Uncertainty Aversion:

$\forall\ acts\ f, g \in \Lambda\ and\ \alpha \in [0,1]: f \sim g\ implies\ that\ \alpha f + (1-\alpha)g \succcurlyeq f$

(21) General complete Maxmin Expected Utility function:

$$W(\mathcal{F}(\cdot)) = \alpha * \min_{\mu \in \mathcal{C}} \int U(\mathcal{F}(\cdot)) d\mu + (1-\alpha) * \max_{\mu \in \mathcal{C}} \int U(\mathcal{F}(\cdot)) d\mu$$

(22) General Rank-Dependent Weight Model:

$\sum_{j=1}^{n} \pi_j U(x_j)$

(23) Certainty Equivalence:

$If\ outcome\ c \in X\ is\ a\ non-risky\ prospect\ in\ the\ set\ of\ non-risky$

$prospects, X, and\ risky\ prospect\ y \in Y, where\ Y\ is\ the\ set\ of\ all\ risky\ prospects,$

$and\ c \sim y\ then\ it\ is\ to\ be\ called\ a\ certainty\ equivalent\ (CE)\ of\ y, or: c = CE(y)$

(24) Rank-Dependent Utility axiom of independence:

$For\ some\ outcome\ x \in X\ and\ some\ probability, \qquad p \in [0,1]$

$If\ y_1 = \{x, p\}\ and\ y_2 = \{x', p\}\ and\ for\ each\ i, i = 1,2, \dots n,\ \exists$:

$c_i = CE\left\{(x_i, x_i'); \left(\frac{1}{2}, \frac{1}{2}\right)\right\}, i \in 1,2 \dots, n, \qquad and$

$x_1^* = CE(y_1), x_2^* = CE(y_2).\ Then$:

$\{c; p\} \sim \left\{(x_1^*, x_2^*), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$

(25) Assumptions 1 and 2 from Rank-Dependent Utility:

(a.) $A1: If\ x_1, \dots, x_n \in X,\ x_n P_{x_{n-1}} P, \dots, P_{x_1}\ and\ \sum_i p_i = 1, then\ \{x; p\} \in Y$

(b.) $A2: If\ y \in Y\ then\ \exists\ some\ x = CE(y) \in X$

(26) Rank-Dependent Utility Theorem 1:

$T1$: *Suppose* X *and* Y *satisfy* $A1$ *and* $A2$, *and* P *satisfies Quiggin's* 4 *axioms including RDU Independence*:

Then $\exists$ *a* [*utility*] *function* $V: Y \rightarrow \mathcal{R}^n$, *st*:

i: $V(y) \geq V(y'), iff\ y \succcurlyeq y'$

ii: $V(x;p) = \sum_i h_i(p)U(x_i)$ *for some functions* U *and* h,

where $h(1) = 1$, *and* $h\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

If the functions V *and* V' *satisfy* i *and* ii, *then* $\exists$ *constants* a *and* b, *with* $a > 0$, *st*:

iii: $V'(y) = aV(y) + b$

(27) Full Rank-Dependent Utility model:[76]

(a.) $RDU(p_1x_1; \dots; p_nx_n) = \sum_{j=1}^{n} \pi_j U(x_j)$

(b.) Decision Weights of the Rank-Dependent Utility Model:

$\pi_j = \omega(p_1 + \dots + p_j) - \omega(p_1 + \dots + p_{j-1})$

(28) Dual Independence:

$G \succcurlyeq G'$, *implies* $aG \boxplus (1-a)H \succcurlyeq aG' \boxplus (1-a)H$

(29) Decumulative Distribution Function:

$G(v) = \Pr\{v > t\}$ *and* $0 \leq t \leq 1$

(30) Mixture Operator ($\boxplus$):

$aG \boxplus (1-a)H = (aG^{-1} + (1-a)H^{-1})^{-1}$

[76] Note that the probabilities do not occur within the model because they are accounted for in the Decision Weights.

(31) Yaari's Representative Theorem:

$$u \succcurlyeq v \Leftrightarrow \int_0^1 f\big(G_u(t)\big)dt \geq \int_0^1 f\big(G_v(t)\big), where\ t\ is\ an\ outcome$$

(32) Prelec's weight Function:

$$g(p) = e^{-(-lnp)^{\gamma}}$$

(33) Generalized Suhonen, Sasstamoinen, and Linden model:

$$V(P) = u(r_0) + \sum_{j=1}^{n} w\big(p_j\big)[u\big(r_j\big) - u\big(r_{j-1}\big)]$$

(34) Uncertainty model from Suhonen, Sasstamoinen, and Linden:

$$V(P) = r_0 + \sum_{j=1}^{n} U\big(p_j\big)[r_j - r_{j-1}]$$

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